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ACOUSTIC COUPLING IN COMBUSTION INSTABILITY

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We shall consider the acoustic instability of premixed combustion in a duct. Our work focuses on the coupling between the spontaneous acoustic waves and the flame front. Using large-activation-energy asymptotic methods, the flame front is described as a discontinuity separating the burnt and unburnt mixtures. The flame front is dynamically coupled with the hydrodynamic and acoustic fields as shown in recent work by Wu et al.. By assuming a linear hydrodynamic field, we obtain an improved Michelson-Sivashinsky flame equation, which includes the back action of the acoustic field on the flame. The steady solutions of our equation are also steady solutions of the classic Michelson-Sivashinsky equation. Subsequently, we shall perturb these steady solutions and study their stability. In particular, we will emphasise the fact that the unsteady perturbation must consist of spontaneous acoustic waves, and that, when these are taken into account, the steady solutions are always unstable. This is an important result given that previous authors have shown that steady solutions of the Michelson-Sivashinsky equation could be stable when the acoustic perturbations are artificially excluded.

1. Introduction

We consider the problem of pre-mixed combustion in a duct (see Figure 1). We consider the combustion as a one-step irreversible chemical reaction, where the fuel is the deficient reactant. The mixture is assumed to obey the state equation for a perfect gas and the fluid is assumed to be Newtonian.

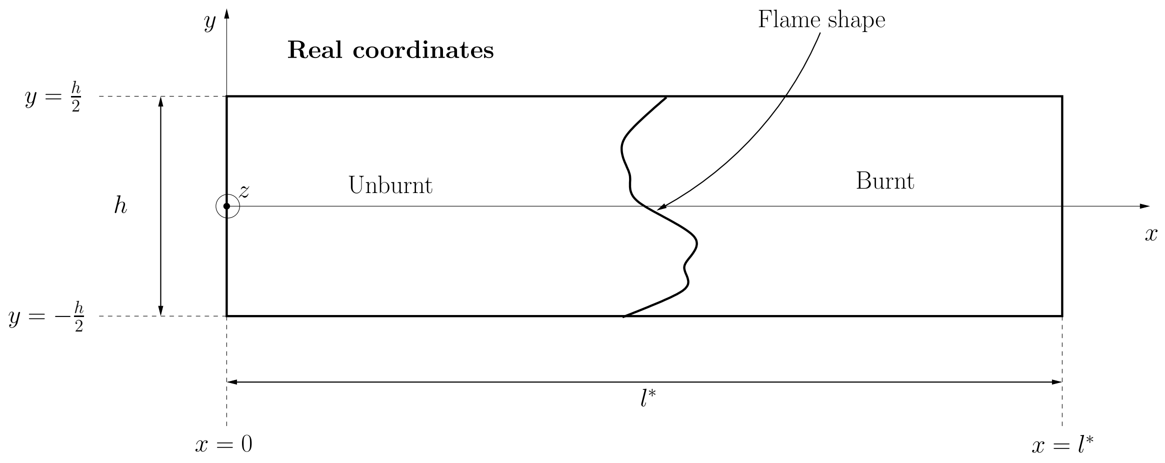

Figure 1. Geometry of the problem

Following [1], we consider five governing equations, namely the equations of conservation of mass, momentum and energy, as well as the state equation and the transport equation governing the diffusion of chemical species. In order to simplify the problem, two classic combustion assumptions are made: we consider that the bulk viscosity can be neglected (Stokes' hypothesis) and we assume that the product of density and the mass diffusion coefficient is constant. The problem is then non-dimensionalised. The space variables are scaled by $h^* = h/2\pi$, where h is the width of the duct. The density and the temperature are scaled by $\rho_{-\infty}$ and $\theta_{-\infty}$ respectively, where $\rho_{-\infty}$ and $\theta_{-\infty}$ are the density and temperature in the fresh mixture (far away from the flame). The velocities are scaled by U_L, the mean propagating speed of the flame, while the scaling of other quantities arise "naturally". Figure 2 shows the non-dimensional geometry.

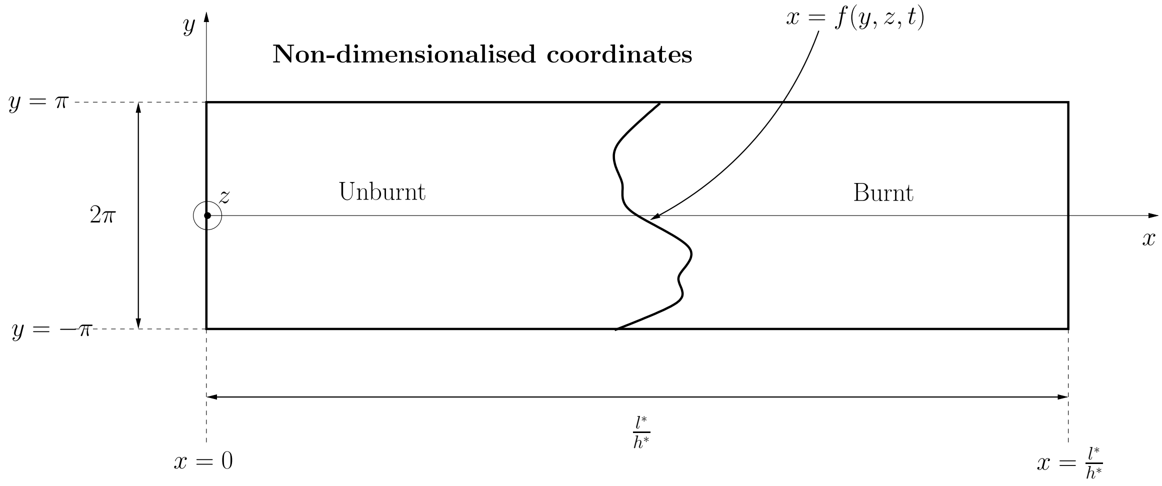

Figure 2. Non-dimensionalised geometry of the problem

This process of non-dimensionalisation leads to a system of five governing equations (see [4]), in which δ represents the flame thickness, Pr is the Prandtl number, Le is the Lewis number, Y is the mass fraction of fuel, G represents the gravity term, Ω is the Arrhenius term, which is proportional to $\delta^{-1}e^{-\beta/\theta}$ (β being the activation energy), γ is the adiabatic index, M is the Mach number and q is the heat release. The flame front is defined as a flame discontinuity in the hydrodynamic field by $x = f(y, z, t)$ (see Figure 2). In order to track this flame front, we perform a change of variable to express the problem in the flame frame of reference. We pass from the variables (x, y, z, t) to the variables (ξ, η, ζ, τ), where $\xi = x - f(x, y, z, t), \eta = y, \zeta = z$ and $\tau = t$. By expressing the velocity field as $\vec{u} = u\overrightarrow{e_\xi} + \vec{v}$, considering a reduced gradient $\tilde{\nabla} = (\partial/\partial\eta \quad \partial/\partial\zeta)^T$ and a new variable $s = u - \partial f/\partial\tau - \vec{v}\cdot\tilde{\nabla}f$, one obtains a new system of governing equations in the flame frame of reference. The resulting geometry is shown in Figure 3.

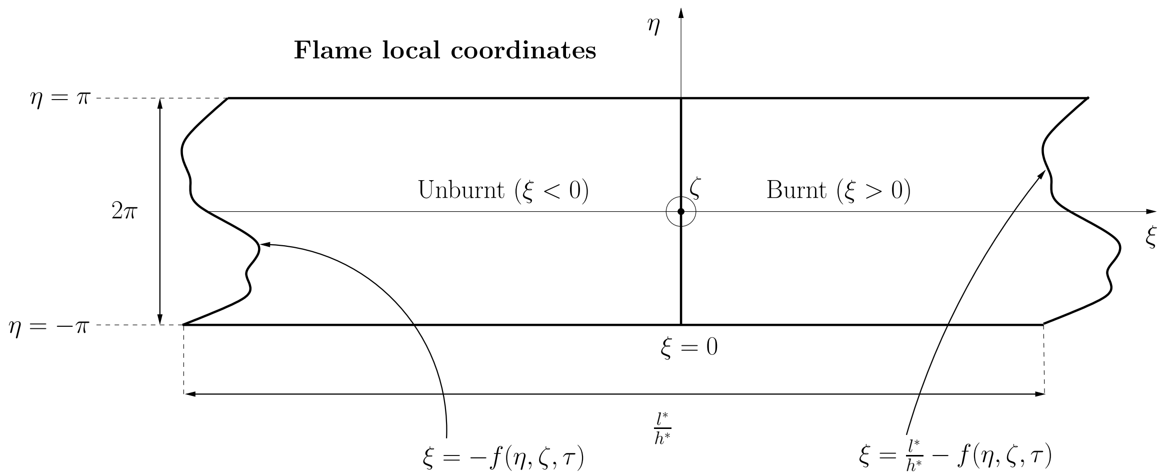

Figure 3. Geometry with a flame frame of reference

2. Asymptotic analysis

In this study, we make three main asymptotic assumptions: a large activation energy ($\beta \gg 1$), a thin flame ($\delta \ll 1$) and a low Mach number ($M \ll 1$). This results in four asymptotic regions described in Figure 4: the reaction zone where $\xi = O(\delta/\beta)$, the pre-heated zone, where $\xi = O(\delta)$, the hydrodynamic zone where $\xi = O(1)$ and the acoustic zone where $\xi = O(1/M)$. These regions are consistent with our assumptions since we have $\delta/\beta \ll \delta \ll 1 \ll 1/M$.

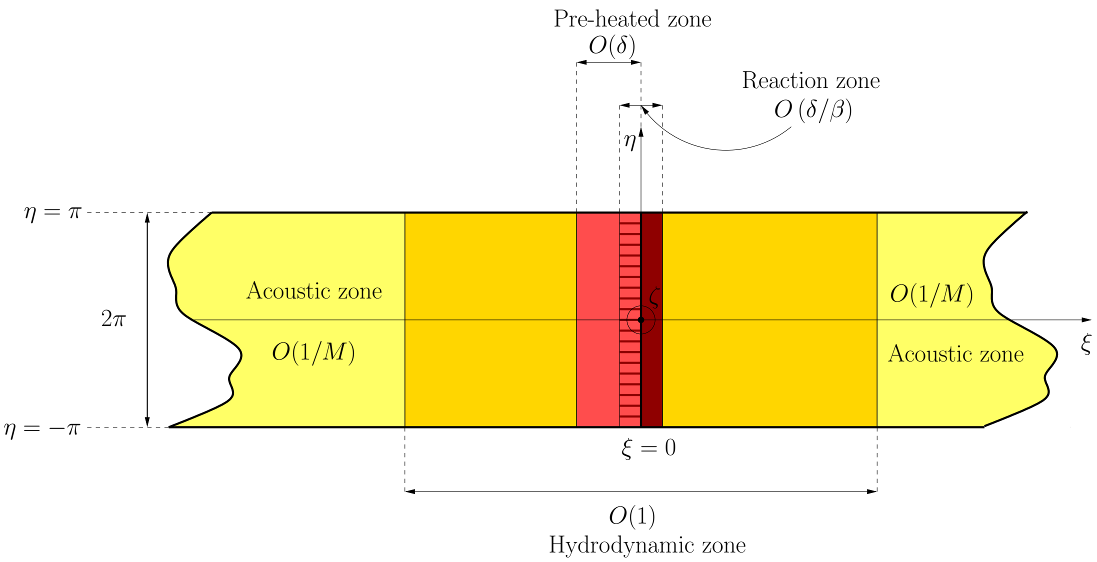

Figure 4. Different asymptotic zones

2.1 Preliminary work in the hydrodynamic zone

In order to capture the hydrodynamic behaviour of the field, let us write a δ-expansion of the main variables:

$$(\rho, \theta, \vec{v}, f, p, Y, s) \quad = \quad (R_0, \Theta, u_0, \vec{v}_0, f_0, p_0, Y_0, s_0) + O(\delta) \tag{1}$$

Inputting (1) into the flame frame of reference governing equations, one obtains that R_0 and Θ should be constant and satisfy $R_0\Theta = 1$. Moreover, according to [1], the flame front is described by the following flame equation:

$$\frac{\partial f_0}{\partial t} \quad = \quad u_0(0^-, \eta, \zeta, \tau) - \vec{v}_0(0^-, \eta, \zeta, \tau) \cdot \tilde{\nabla} f_0 - \left[1 + \left(\tilde{\nabla} f_0\right)^2\right]^{1/2} + \delta\nu \tilde{\nabla}^2 f_0,$$

and the hydrodynamic field is subject to the following jump conditions across the flame front:

$$[u_0] = q\left[1 + \left(\tilde{\nabla} f_0\right)^2\right]^{-1/2}, \quad [\vec{v}_0] = -q\left(\tilde{\nabla} f_0\right)\left[1 + \left(\tilde{\nabla} f_0\right)^2\right]^{-1/2}, \quad [p_0] = -q.$$

2.2 The acoustic zone

In order to capture the acoustic behaviour of the field, one should first perform the change of variable $\tilde{\xi} = M\xi$ in the governing system. We then write a M-expansion of the main variables. These expansions arise naturally from balancing the governing equations:

$$\begin{cases} u = U_\pm + u_a\left(\tilde{\xi}, \tau\right) + O(M) \\ \rho = R_0 + M\rho_a\left(\tilde{\xi}, \tau\right) + O(M^2) \end{cases} \qquad \begin{cases} p = M^{-1}\left\{p_a\left(\tilde{\xi}, \tau\right) - R_0 G\tilde{\xi}\right\} + O(1) \\ \theta = \Theta + M\theta_a(\tilde{\xi}, \tau) + O(M^2) \end{cases}. \tag{2}$$

Inputting (2) into the governing system and eliminating the variables θ_a and ρ_a leads to the equations governing the acoustic fluctuations as well as their jump[1] across the hydrodynamic zone:

$$\left\{\begin{array}{l} \frac{\partial p_a}{\partial \tau}+\frac{\partial u_a}{\partial \tilde{\xi}}=0 \\ -\frac{\partial^2 p_a}{\partial \tilde{\xi}^2}+R_0\frac{\partial^2 p_a}{\partial \tau^2}=0 \end{array}\right. , \quad \left\{\begin{array}{l} [p_a]=0 \\ [u_a]=\mathcal{J}_a(\tau)=q\left\{\overline{\left[1+\left(\tilde{\nabla} f_0\right)^2\right]^{1/2}}-1\right\} \end{array}\right. , \tag{3}$$

where the notation represents a space average.

2.3 More on the hydrodynamic zone

In order to use our knowledge of the acoustic zone to describe the hydrodynamic zone, let us define the new hydrodynamic variables $U_0, \overrightarrow{V_0}, S_0, P_0$ and F_0 by

$$\left\{\begin{array}{l} u_0=U_\pm+u_a\left(0^\pm,\tau\right)+U_0 \\ s_0=S_0+1+\left(q+\mathcal{J}_a(\tau)\,\mathbb{H}(\xi)\right. \end{array}\right. , \quad \left\{\begin{array}{l} \overrightarrow{v_0}=\overrightarrow{V_0} \\ f_0=F_a+F_0 \end{array}\right. \tag{4}$$

and $p_0=M^{-1}p_a(0,\tau)+P_\pm+\left(\frac{\partial p_a}{\partial \tilde{\xi}}(0^\pm,\tau)-R_0G\right)(F_0+\xi)+P_0$, where $\mathbb{H}$ is the classic Heaviside function. Hence the flame equation describing F_0 becomes

$$\frac{\partial F_0}{\partial \tau} = U_0(0^-,\eta,\zeta)-\overrightarrow{V_0}\left(0^-,\eta,\zeta\right)\cdot\tilde{\nabla}F_0-\left\{\left(1+\left(\tilde{\nabla}F_0\right)^2\right)^{1/2}-1\right\}+\delta\nu\tilde{\nabla}^2F_0 \tag{5}$$

Note that both the acoustic and hydrodynamic expansion of the pressure have been chosen carefully so that gravity does not appear in the governing equations. Using (4), one obtains the equations governing the hydrodynamic zone subject to the following jump conditions:

$$\left\{\begin{array}{lcl} [U_0] &=& q\left[1+\left(\tilde{\nabla}F_0\right)^2\right]^{-1/2}-q-\mathcal{J}_a(\tau) \\ [\vec{V_0}] &=& -q\left(\tilde{\nabla}F_0\right)\left[1+\left(\tilde{\nabla}F_0\right)^2\right]^{-1/2} \\ [P_0] &=& -\left(\mathcal{L}_a(\tau)+\frac{qG}{1+q}\right)F_0 \end{array}\right. , \tag{6}$$

where $\mathcal{L}_a(\tau)$ represents the jump of $\partial p_a/\partial\tilde{\xi}$ across the hydrodynamic zone. The presence of $\mathcal{J}_a$ and $\mathcal{L}_a$ in the hydrodynamic equations is clear evidence of acoustic coupling. This acoustic coupling has been studied in [2], where an artificial cosine flame was perturbed and shown to be unstable. In Section 3, we shall find explicit steady flames and study their stability in Section 4.

3. Steady state solutions to the flame equation

In order to find steady solutions of the flame equations, we shall now drop the index $_0$, consider the problem as a 2D problem in space (the space variables are reduced to (ξ,η), and $\vec{V}=V$) and assume that the velocity field is steady. Moreover we consider that there are no acoustic fluctuations and that the hydrodynamic field is small. Under these assumptions, the equations describing the hydrodynamic zone simplify significantly and become:

$$\left\{\begin{array}{l} \frac{\partial U}{\partial \xi}=-\frac{\partial V}{\partial \eta} \\ \frac{\partial U}{\partial \xi}=-\frac{\partial P}{\partial \xi} \\ \frac{\partial V}{\partial \xi}=-\frac{\partial P}{\partial \eta} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} [U]=0 \\ [V]=-q\left(\frac{\partial F}{\partial \eta}\right) \\ [P]=\frac{-qG}{1+q}F \end{array}\right. . \tag{7}$$

[1]Actually, in order to obtain $[u_a]$, more work is required. However for brevity, this will not be presented here.

This system is then solved using Fourier transforms in the η direction (denoted by $\widehat{\ }$). Hence all the Fourier transforms of the main variables can be expressed in terms of $\hat{F}$, the Fourier transform of F. The partially linearised flame equation then becomes

$$\begin{cases} \frac{\partial F}{\partial \tau} = U(0^-, \eta) - \frac{1}{2}\left(\frac{\partial F}{\partial \eta}\right)^2 + \delta\nu \frac{\partial^2 F}{\partial \eta^2} \\ \hat{U}\left(0^-, k\right) = \frac{q\hat{F}}{2}\left(|k| - \frac{G}{1+q}\right) \end{cases} . \tag{8}$$

At this stage, one should notice the similarities between (8) and the well-known Michelson-Sivashinsky (M-S) equation studied in detail in [3]. The M-S equation is the following:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \frac{1}{2} I(\varphi; \eta) + \frac{1}{\gamma}\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial \eta}\right)^2 \\ \widehat{I(\varphi; \eta)}(k, t) = |k|\hat{\varphi}(k, t) \end{cases} . \tag{9}$$

The similarity between (8) and (9) lies in the fact that if the gravity is neglected ($G = 0$), then the two equations are equivalent via the change of variable $t = q\tau$, $\varphi = -F/q$ and $\delta\nu = q/\gamma$. In [3], a rigorous theory about the M-S equation is presented. Let us summarise it briefly. The M-S equation admits some N-poles solutions taking the form

$$\varphi(\eta, t) \quad = \quad \varphi_0(t) + \frac{2}{\gamma}\sum_{n=1}^{N} \ln\left[\frac{1}{2}\left\{\cosh(y_n(t)) - \cos(r\eta - x_n(t))\right\}\right], \tag{10}$$

with N pairs of complex conjugate poles $z_n(t) = x_n(t) + iz_n(t)$. The maximum value of poles that may exist depends on γ and is given by $N_{\max} = \mathrm{Int}[\gamma/2]$. Naturally, the poles tend to align vertically. A coalescent pole solution is a pole solution such that the poles are aligned vertically and a steady coalescent pole solution is a coalescent pole solution such that the location of the poles is time-independent. One of the results in [3] is that for a given pair (γ, N), there exists a steady coalescent pole solution only if $N \leqslant N_0(\gamma)$, where $N_0(\gamma) = \mathrm{Int}[\gamma/4 + 1/2]$. In addition, for a given $N \leqslant N_0(\gamma)$, this solution is unique and denoted $\varphi_N(\gamma, \eta, t)$. The one and two-pole solutions are known analytically. Hence this is a good way to verify which steady solutions are captured numerically. In order to do so, we express F as a truncated Fourier series. The comparison between theory and numerics is shown in Figure 5.

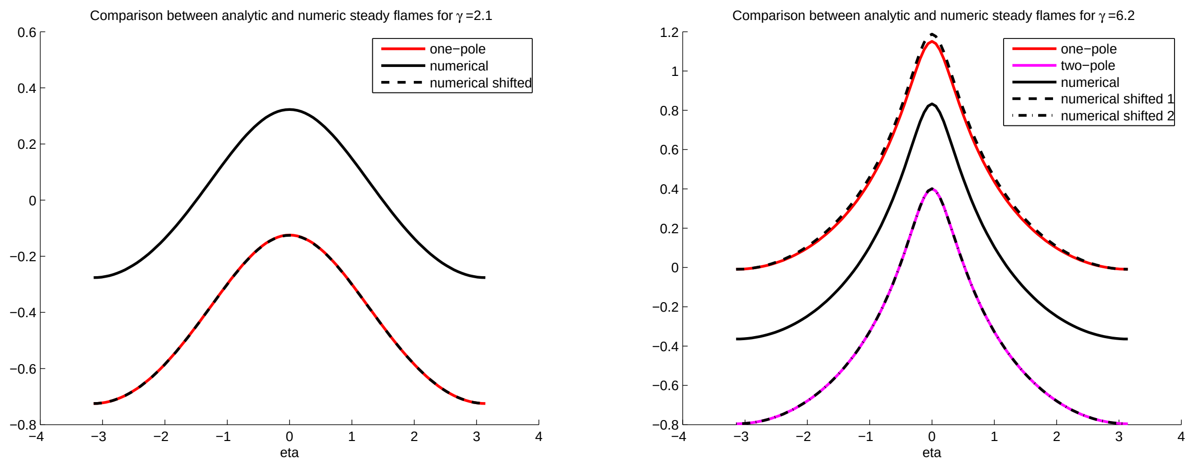

Figure 5. Comparison between the theoretical steady solutions and the numerical steady solutions for two values of γ, $\gamma = 2.1$ (left) and $\gamma = 6.2$ (right)

From Figure 5, it is clear that in the case of $\gamma = 2.1$ ($N_0 = 1$), we have found numerically the unique one-pole steady solution, while in the case $\gamma = 6.2$ ($N_0 = 2$), we have found the unique two-pole solution.

4. Stability study

Now that we have found some steady solutions, we shall study their linear stability. In order to do so, let us perturb the steady state, by writing, for a small value of ε:

$$\{U, V, P, F, u_a, p_a\} \quad = \quad \{U_s, V_s, P_s, F_s, (u_a)_s, (p_a)_s\} + \varepsilon \left\{\tilde{U}, \tilde{V}, \tilde{P}, \tilde{F}, \widetilde{u_a}, \widetilde{p_a}\right\}, \tag{11}$$

where the subscript $_s$ represents the steady state and the superscript $\tilde{}$ represents the perturbation. Inputting (11) into the hydrodynamic equations, considering that the product of two quantities is small, one obtains the perturbed equations and jump conditions in the hydrodynamic and acoustic zones. In order that the acoustic system be closed, one should impose some boundary conditions (b.c.) at the extremities of the duct. In order to translate these conditions to the flame frame of reference, it is convenient to define a position average of the flame front. This position average, denoted σ, is defined in Figure 6. It is such that when the combustion starts (i.e. none of the mixture is yet burnt) we have $\sigma = 1$ and when it stops (i.e. all the mixture has been burnt) we have $\sigma = 0$.

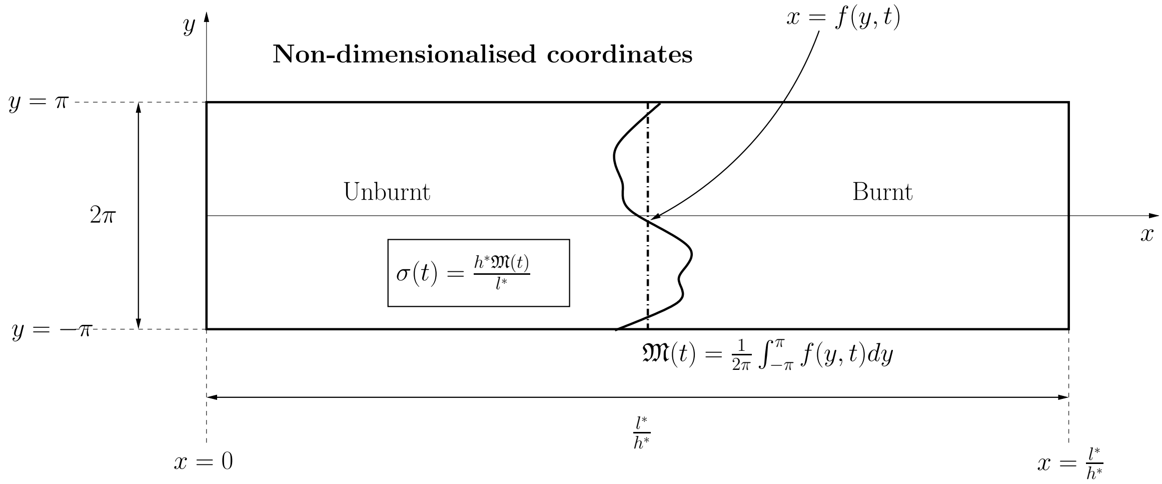

Figure 6. Definition of σ

In order to obtain results concerning the stability of our steady state solutions, let us assume that the perturbations are harmonic in time, that is that we can write

$$\left\{\tilde{U}, \tilde{V}, \tilde{P}, \widetilde{u_a}, \widetilde{p_a}\right\} \quad = \quad 2\Re\left(\{\mathfrak{U}, \mathfrak{V}, \mathfrak{P}, \mathfrak{F}, \mathfrak{u}_\mathfrak{a}, \mathfrak{p}_\mathfrak{a}\} \exp\left(i\omega\tau\right)\right). \tag{12}$$

Hence if by some method, we manage to find ω and if $\Im(\omega) < 0$, then our solution is unstable. The now closed acoustic system can be solved using (12) and Fourier transforms w.r.t. η. It leads to the following acoustic dispersion relation:

$$\Delta_s(\omega, \sigma) = \frac{1}{\sqrt{1+q}} \tan(\omega\sigma L) \tan\left(\frac{\omega(1-\sigma)L}{\sqrt{1+q}}\right) - 1 = 0. \tag{13}$$

The hydrodynamic system can also be solved, hence the linearised perturbed flame equation becomes

$$\left\{\begin{array}{l} \frac{\partial \tilde{F}}{\partial \tau} = \tilde{U}(0^-, \eta) - \frac{\partial F_s}{\partial \eta}\frac{\partial \tilde{F}}{\partial \eta} + \delta\nu \frac{\partial^2 \tilde{F}}{\partial \eta^2} \\ \hat{\mathfrak{U}}(0^-, k) = \frac{q\left(|k|^2 - \frac{|k|G}{1+q}\right)}{\left(\frac{i\omega}{1+q} + |k|\right) + (|k| + i\omega)} \hat{\mathfrak{F}} + \frac{\frac{i\omega|k|}{1+q}\left(1 - \frac{q}{\sqrt{1+q}} \frac{\tan\left(\frac{\omega(\sigma-1)L}{\sqrt{1+q}}\right)\tan(\omega\sigma L)}{\Delta_s(\omega,\sigma)}\right)}{\left(\frac{i\omega}{1+q} + |k|\right) + (|k| + i\omega)} \mathfrak{I}_\mathfrak{a} \widehat{F_s} \end{array}\right. . \tag{14}$$

Again, one can compare this equation to the linearised perturbed M-S equations studied in [3] that is:

$$\left\{ \begin{array}{l} \frac{\partial \psi}{\partial t} = \frac{1}{2} I\left(\psi;\eta\right) + \frac{1}{\gamma}\frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial \varphi_N}{\partial \eta}\frac{\partial \psi}{\partial \eta} \\ \widehat{I\left\{\psi;\eta\right\}}(k,t) = |k|\hat{\psi}(k,t) \end{array} \right. . \tag{15}$$

In order to obtain such a perturbed M-S equation, previous authors had to make three main assumptions. The first one is that $\frac{\partial \tilde{U}}{\partial \tau} = \frac{\partial \tilde{V}}{\partial \tau} = 0$, the second one is to neglect acoustic perturbations and the last one is to neglect gravity. If we make these assumptions, then equation (14) becomes the linearised perturbed M-S (15) when the changes of variable $t = q\tau$, $\psi = -\tilde{F}/q$ and $\delta\nu = q/\gamma$ are made. The theory developed in [3] is summarised in Figure 7.

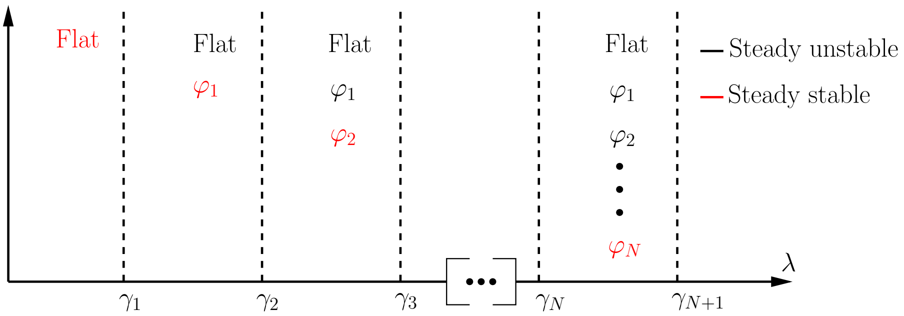

Figure 7. Summary of stability results about N-pole solutions

When making the aforementioned assumptions, expressing $\tilde{F}$ as a Fourier series, we can confirm these results numerically (the flame equation is changed into a linear eigenvalue problem in ω). However, when we are not making these simplification assumptions, the problem becomes more difficult to solve. Indeed, writing $\tilde{F}$ as a Fourier series in η leads to a non-linear eigenvalue problem that is σ-dependent. In order to solve it, for each value of σ, we linearise the eigenvalue problem around a characteristic frequency, ω_k, chosen to be the smallest root of $\Delta_s(\omega, \sigma)$. Consequently we end up with a linear eigenvalue problem that can be solved for each value of σ. For the two values of γ used in Figure 5, Figure 8 shows the least stable growth factor for each value of σ. The result is important since Figure 8 clearly shows that the field is unstable for all values of σ. Hence whether one considers the acoustic when studying premixed combustion makes a huge difference.

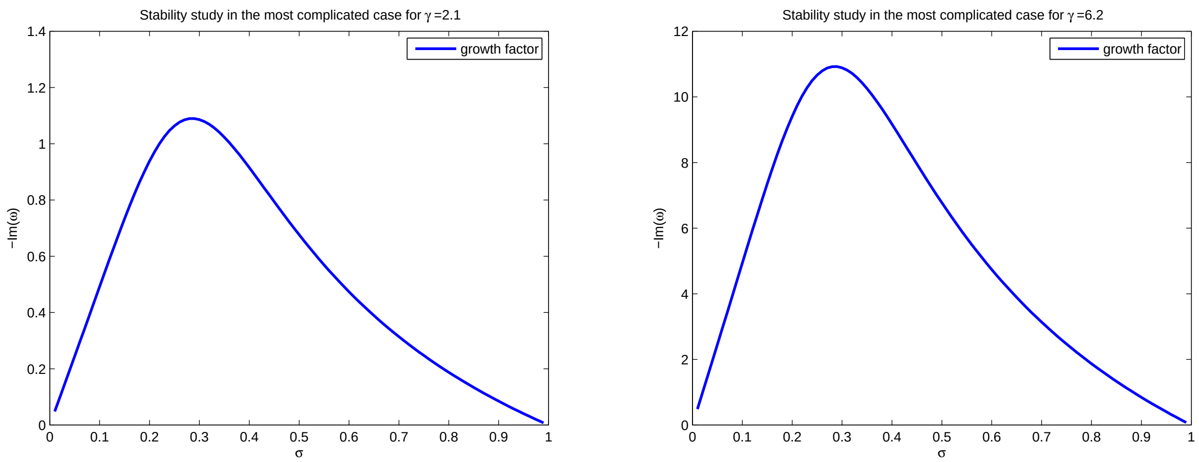

Figure 8. Plot of the growth factor versus σ for $\gamma = 2.1$ (left) and $\gamma = 6.2$ (right)

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