MODERN THEORY OF ORTHOGONAL POLYNOMIALS AND THEIR LINK WITH JULIA SETS

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Abstract. This essay develops the basic generic theory of orthogonal polynomials on the real axis, in order to present Favard's theorem. A generalization of the concept of orthogonal polynomials on a complex contour is also developed. This essay emphasizes the family formed by the iterates of a complex polynomial T. This leads to an orthogonal polynomial sequence which is orthogonal "on" the Julia set generated by T.

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Part 1. General theory of orthogonal polynomials on the real axis

1. Introduction

The concept of orthogonal polynomials on the real axis is often introduced in undergraduate courses in mathematics, but mainly in the form of particular examples like the Legendre polynomials. Nevertheless a much more general theory has been developed in the 20th century. This theory emerged after the discovery of particular examples which is why I will start by giving some of those examples and then try to show how these examples can lead to a general theory. I will expose some of the common properties that are verified by every family of orthogonal polynomials up to the Favard's theorem that gives a method to construct such a family.

Then I focus on a generalization of the concept of orthogonality of polynomials on a complex contour in order to obtain the main result of this essay which is: geven any complex polynomial its iterates lead to a subsequence of a family of orthogonal polynomials. In order to prove this, we will have to go through the concept of Julia sets and consequently the concept of fractals.

2. Examples leading to the first definitions

2.1. Examples.

Example 2.1. ***Legendre polynomials***

Let (P_n)be the family of polynomials defined by

$$P_n(x) = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left(\left(1-x^2\right)^n\right)$$

It can be proved than each P_n is a polynomial of degree n and that they verify the following relation:

$$\forall (m,n)\,\epsilon\mathbb{N}\times\mathbb{N}, \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \frac{2}{2n+1}\delta_{m,n}$$

where $\delta_{m,n}$ is the classic symbol of Kronecker: $\delta_{m,n} = \begin{cases} 1\,if\,m=n \\ 0\,if\,m\neq n \end{cases}$. It can therefore be said that the Legendre polynomials are orthogonal on the interval $\left[-1,1\right]$.

Example 2.2. ***Tchebychev polynomials***

Let (T_n)be the family of polynomials defined by

$$T_n(x) = \cos\left(n.\arccos(x)\right)$$

It can be proved that each T_n is a polynomial of degree n and that they verify the following relation:

$$\forall (m,n)\,\epsilon\mathbb{N}^*\times\mathbb{N}^*, \int_{-1}^{1} T_m(x)\,T_n(x)\,.\frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}\delta_{m,n}$$

This family of polynomials is therefore called an orthogonal family of polynomials on the interval $[-1,1]$ with respect to the weigth function $x\mapsto\frac{1}{\sqrt{1-x^2}}$.

Example 2.3. ***Laguerre Polynomials***
Let (L_n)be the family of polynomials defined by

$$L_n = \frac{1}{n!}\exp(x)\frac{d^n}{dx^n}(x^n\exp(-x))$$

It can be proved that each L_n is a polynomial of degree n and that they verify the following relation:

$$\forall (m,n)\,\epsilon\mathbb{N}\times\mathbb{N}, \int_0^\infty L_m(x)\,L_n(x)\exp(-x)\,dx = \delta_{m,n}$$

It is therefore said that the Laguerre polynomials form an orthogonal family of polynomials on $\mathbb{R}^+$ with respect to the weight function $x \mapsto \exp(-x)$. In this case , because of the "normalisation" of the expression, it is said that the Laguerre polynomials are orthonormal.

Example 2.4. ***Hermite Polynomials***
Let (H_n)be the family of polynomials defined by

$$H_n(x) = (-1)^n\exp\left(x^2\right)\frac{d^n}{dx^n}\left(\exp\left(-x^2\right)\right)$$

It can be proved that each H_n is a polynomial of degree n and that the H_n's verify the following relation:

$$\forall (m,n)\,\epsilon\mathbb{N}\times\mathbb{N}, \int_{-\infty}^\infty H_m(x)\,H_n(x)\exp\left(-x^2\right)dx = \sqrt{\pi}2^n\,(n!)\,\delta_{m,n}$$

It is therefore said that the Hermite polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function $x \mapsto \exp\left(-x^2\right)$.

2.2. **Observations.**

In all the examples above, the polynomials are linked to a weight function and a particular interval of $\mathbb{R}$ on which the weight function is integrable and ≥ 0. These 2 characteristics are enough to create an inner product for which the family is orthogonal. Let w be the weight function and $[a,b]$ the interval, then the inner product $<,>$ can be defined as follows:

$$\forall(f,g)\epsilon C\left[a,b\right], < f,g >= \int_a^b f(x)\,g(x)\,w(x)\,dx$$

These observations naturally lead to our first formal definitions.

2.3. **Formal definitions.**

Definition 2.5. We call ***weight function*** on $\left[a,b\right]$, a function w which verifies the following properties:

- w is integrable on $\left[a,b\right]$
- $\forall x\epsilon\left[a,b\right], w(x) \geq 0$
- $w(x) > 0$ on a subset of $\left[a,b\right]$ of positive Lebesgue measure
- $\forall n\epsilon\mathbb{N}$, the "moments" $\mu_n = \int_a^b x^n w(x)\,dx < \infty$

Remark 2.6. The third property is here to ensure that $\int_a^b w(x)\,dx > 0$.The fourth property is going to be needed further for the case of an unbounded $\left[a,b\right]$.

Definition 2.7. We call an ***orthogonal polynomial sequence*** (OPS) with respect to the weight function w on $[a,b]$, a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ such that:

- $\forall n \epsilon \mathbb{N}, \deg(P_n) = n$
- $\forall (m,n) \epsilon \mathbb{N} \times \mathbb{N},\ m \neq n \Rightarrow \int_a^b P_m(x) P_n(x) w(x) dx = 0$
- $\forall n \epsilon \mathbb{N}, \int_a^b P_n(x)^2 w(x) dx \neq 0$

Remark 2.8. Clearly, the 4 examples above, because of their mentioned properties, are OPS for their respective weight function.

Notation 2.9. Given a weight function w and an interval $[a,b]$ we note:

$$\mathcal{L}[f] = \int_a^b f(x) w(x) dx$$

for any function f integrable on $[a,b]$. Therefore the "moments" condition of the definition 2.7 can be noted $\forall n \epsilon \mathbb{N}, \mathcal{L}[x^n] < \infty$ and the orthogonality property of the definition 2.5 can be noted

$$\forall (n,m) \epsilon \mathbb{N} \times \mathbb{N}, m \neq n \Rightarrow \mathcal{L}[P_m(X) P_n(X)] = 0$$

The introduction of this notation will allow us to have much more general definitions for the OPS based on the fact that, given an arbitrary sequence of complex numbers $\{\mu_n\}_{n=0}^{\infty}$, we can define uniquely a linear functional $\mathcal{L}$ on the vector space of the polynomials of one real variable $\mathbb{R}[X]$. This naturally leads to the following definition.

Definition 2.10. Let $\{\mu_n\}_{n=0}^{\infty} \epsilon \mathbb{C}^{\mathbb{N}}$. We call the ***moment functional*** determined by the ***moment sequence*** $\{\mu_n\}$, the linear functional $\mathcal{L} : \mathbb{R}[\mathbb{X}] \longrightarrow \mathbb{C}$ such that $\forall n \epsilon \mathbb{N}, \mathcal{L}[X^n] = \mu_n$

This introduction of the concept of moment functional will naturally lead to a new approach to the OPS by giving a new definition.

Definition 2.11. A sequence $(P_n) \epsilon \mathbb{R}[X]^{\mathbb{N}}$ is called an OPS with respect to the moment functional $\mathcal{L}$, if it verifies the 3 following conditions for any $m, n \geq 0$:

- $\deg(P_n) = n$
- $\mathcal{L}[P_m(X) P_n(X)] = 0$ for $m \neq n$
- $\mathcal{L}\left[(P_n(X))^2\right] \neq 0$

3. General results on real orthogonal polynomials[1]

3.1. Basic results.

Lemma 3.1.

Hypothesis: *$\mathcal{L}$ is a moment functional, $(P_n) \epsilon \mathbb{R}[X]^{\mathbb{N}}$*

Conclusions: *Those 3 properties are equivalent to each other*

(1) $\{P_n(X)\}$ *is an OPS for* $\mathcal{L}$

(2) $\forall n \epsilon \mathbb{N}, \forall \pi \epsilon \mathbb{R}_m[X], \mathcal{L}[\pi(X) P_n(X)] \begin{cases} = 0 \, if \, m < n \\ \neq 0 \, if \, m = n \end{cases}$

(3) $\exists (K_n) \epsilon \mathbb{C}^{\mathbb{N}}$, *such that* $\forall (n,m) \epsilon \mathbb{N} \times \mathbb{N}, \mathcal{L}[X^m P_n(X)] = K_n \delta_{m,n}, K_n \neq 0$

[1]The results presented in sections 3 and 4 are mainly taken from [2], but the proofs are more detailed.

Proof. Let us assume the conclusion (1). Since $\deg(P_n) = n$, $\{P_0, P_1, \dots, P_m\}$ forms a basis of $\mathbb{R}_m[X]$. Therefore $\forall \pi \epsilon \mathbb{R}_m[X], \exists (c_k)_{k=0}^m$such that $\pi(X) = \sum_{k=0}^m c_k P_k(X)$. Let $\pi \epsilon \mathbb{R}_m[X]$

.if $m < n$ then $\mathcal{L}[\pi(X)P_n(X)] = \sum_{k=0}^m c_k \mathcal{L}[P_k(X)P_n(X)]$ (because $\mathcal{L}$ is linear)
$= 0$ (because (Pn) is an OPS for $\mathcal{L}$ and $m < n$)

.if $m = n$ then
$\mathcal{L}[\pi(X)P_n(X)] = \sum_{k=0}^n c_k \mathcal{L}[P_k(X)P_n(X)] = c_n \mathcal{L}[P_n(X)^2] \neq 0$ (because (P_n) is an OPS for $\mathcal{L}$)

So we have proved that (1)$\Rightarrow$(2).But (2)$\Rightarrow$(3), because 0 can always be written $K_n \cdot 0$ so by taking $K_n = \mathcal{L}[X^n P_n(X)]$ we have the property (3). The property (3) $\Rightarrow$

$$\mathcal{L}[P_m(X)P_n(X)] = \sum_{k=0}^{m} a_k \mathcal{L}[X^k P_n(X)] = a_n K_n \delta_{m,n}$$

so this exactly leads to the fact that the (P_n) are OPS for $\mathcal{L}$.Therefore (3)$\Rightarrow$(1) and then (1)$\Longleftrightarrow$(2) $\Longleftrightarrow$(3). This finishes the proof. □

Theorem 3.2.
Hypothesis: *(P_n) and (Q_n) are OPS for a given moment functional $\mathcal{L}$*
Conclusion: *$\exists (c_n) \epsilon (\mathbb{C}^*)^{\mathbb{N}}$, such that , $\forall n \epsilon \mathbb{N}, Q_n(X) = c_n P_n(X)$*

Proof. The property (2) of the Lemma 3.1 $\Rightarrow \forall k < n$, $\mathcal{L}[P_k(X)Q_n(X)] = 0$.But (Q_n) being an OPS , $\deg(Q_n) = n = deg(P_n)$ therefore the $\{P_k\}_{k=0}^n$ form a basis of $\mathbb{R}_n[X]$ and Q_n can be written as follow: $Q_n(X) = \sum_{k=0}^n c_k P_k(X)$. Therefore $\mathcal{L}[P_k(X)Q_n(X)] = c_k \mathcal{L}[P_k(X)^2]$ i.e. $c_k = \frac{\mathcal{L}[P_k(X)Q_n(X)]}{\mathcal{L}[P_k(X)^2]}$. Then $Q_n(X) = \sum_{k=0}^n c_k P_k(X) = c_n P_n(X)$ (because if $k \neq n$,$c_k = 0$) and $c_n \neq 0$ because of the definition of an OPS and the property (3) of the Lemma 3.1. This finishes the proof. □

In order to introduce a concept of uniqueness of OPS for a given moment functional, we need the following definition.

Definition 3.3. We say that an OPS (P_n) for the moment functional $\mathcal{L}$ is ***monic*** if the leading coefficient of $P_n(X)$ (coefficient of X^n) is equal to 1.

Remark 3.4. It has to be noticed that at this point, the Theorem 3.2 directly gives the following result: If there exists an OPS for a given moent functional $\mathcal{L}$, then there is a *unique monic OPS* for $\mathcal{L}$.

If existence, we now have a result of uniqueness. It is now natural to focus our interest on the problem of existence.

Theorem 3.5.
Hypothesis: *$\mathcal{L}$ is a moment functional with a moment sequence $\{\mu_n\}$*
Conclusion: *There exists an OPS for $\mathcal{L}$*

$$\Longleftrightarrow \forall n \epsilon \mathbb{N}, \Delta_n = det(\mu_{i+j})_{i,j=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \cdots & \mu_{2n} \end{vmatrix} \neq 0$$

Proof. .If there exist an OPS for $\mathcal{L}$, let call it (P_n), then

$$\exists (K_n)\epsilon(\mathbb{C}^*)^{\mathbb{N}}\, such\, that\, \forall\, (m,n)\,\epsilon\mathbb{N}\times\mathbb{N}, \mathcal{L}[X^m P_n(X)] = K_n\delta_{m,n}$$

And then by writing $P_n(X) = \sum_{k=0}^{n} c_{n,k}X^k$ we have:

$$\forall m \leq n, \sum_{k=0}^{n} c_{n,k}\mathcal{L}[X^{m+k}] = K_n\delta_{m,n}$$

so $\sum_{k=0}^{n} c_{n,k}\mu_{m+k} = K_n\delta_{m,n}$.In other words the system of linear equations

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \cdots & \mu_{2n} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ \vdots \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K_n \end{pmatrix} \tag{3.1}$$

has a unique solution. The solution is unique because if there is an other OPS (Q_n) that verifies the same equality, then by the Theorem 3.2:

$$\exists (c_n)\epsilon(\mathbb{C}^*)^{\mathbb{N}}, \forall n\epsilon\mathbb{N}, Q_n(X) = c_n P_n(X)$$

with

$$c_n = \frac{\mathcal{L}[P_n(X)Q_n(X)]}{\mathcal{L}[P_n(X)^2]} = \frac{c_{n,n}\mathcal{L}[X^n Q_n(X)]}{\mathcal{L}[P_n(X)^2]} = \frac{c_{n,n}K_n}{c_{n,n}K_n} = 1$$

therefore $(Q_n) = (P_n)$. Therefore, $\Delta_n \neq 0$.

.Conversely, if $\Delta_n \neq 0$, then the system (3.1) has a unique solution, and this solution verifies the property (3) of the Lemma 3.1. So this solution is an OPS for $\mathcal{L}$. This finishes the proof. □

Theorem 3.6.

Hypothesis: *(P_n) is an OPS for the moment functional $\mathcal{L}$*

Conclusion: $\forall \pi\epsilon\mathbb{R}_n[X]\diagdown\mathbb{R}_{n-1}[X], \forall n \geq 1, \mathcal{L}[\pi(X)P_n(X)] = \frac{a_n k_n \Delta_n}{\Delta_{n-1}}$ *where a_n and k_n are respectively the leading coefficients of π and P_n*

Proof. Let take $\pi\epsilon\mathbb{R}_n[X]\diagdown\mathbb{R}_{n-1}[X]$ then we can write $\pi(X) = a_n X^n + \gamma(X)$ where $\gamma\epsilon\mathbb{R}_{n-1}[X]$. Therefore,

$$\mathcal{L}[\pi(X)P_n(X)] = a_n\mathcal{L}[X^n P_n(X)] + \mathcal{L}[\gamma(X)P_n(X)]$$

the second term is equal to 0 by the property (3) of the Lemma 3.1. So we have

$$\mathcal{L}[\pi(X)P_n(X)] = a_n\mathcal{L}[X^n P_n(X)] = a_n K_n \tag{3.2}$$

But the Cramer formula applied to the system (3.1) tells us that $\mathrm{c}_{n,n} = K_n\frac{\Delta_{n-1}}{\Delta_n}$ but here $c_{n,n}$ is called k_n . Therefore we have $K_n = k_n\frac{\Delta_n}{\Delta_{n-1}}$. So (3.2) becomes

$$\mathcal{L}[\pi(X)P_n(X)] = a_n k_n\frac{\Delta_n}{\Delta_{n-1}}$$

This finishes the proof. □

Let now focus our attention on the influence of particular properties of the moment functional on the related OPS.

3.2. **Moment functional analysis.**

Definition 3.7. A moment functional $\mathcal{L}$ is called ***positive-definite,*** if

$$\forall \pi \epsilon \mathbb{R}[X], such\, that \left\{ \begin{array}{c} \pi \neq 0_{\mathbb{R}[X]} \\ \forall x \epsilon \mathbb{R}, \pi(x) \geq 0 \end{array} \right. \text{ we have } \mathcal{L}[\pi(X)] > 0$$

Theorem 3.8.

Hypothesis: *$\mathcal{L}$ is a positive definite moment functional*

Conclusion: *$\mathcal{L}$ has real moments and a corresponding OPS of real polynomials*

Proof. **.**For any integer k,the polynomial X^{2k} verifies the conditions that have to be imposed to π in the Definition 3.7 . So by definition of a positive-definite moment functional, we have $\mathcal{L}[X^{2k}] > 0$ i.e $\mu_{2k} > 0$ so the moments of even indice are real and strictly positive. Let us now prove by strong recurrence that for any integer n μ_n is real. For $n = 1$, let us consider the polynomial $(X+1)^2$, it verifies the conditions imposed on π in the Definition 3.7 so

$$\mathcal{L}[(X+1)^k] > 0 \tag{3.3}$$

But $(X+1)^2 = X^2 + 2X + 1$, therefore (3.3) becomes $\mu_0 + 2\mu_1 + \mu_2 > 0$, but $\mu_0$and μ_2 are real so μ_1 has to be real. Let us suppose that for any integer $k \leq 2n$, μ_k is real. Let us now consider the polynomial $(X+1)^{2(n+1)} = \sum_{k=0}^{2n+2} \begin{pmatrix} 2n+2 \\ k \end{pmatrix} X^k$ the same reasoning as for the case $n = 1$ leads to the following relation: $\sum_{k=0}^{2n+2} \begin{pmatrix} 2n+2 \\ k \end{pmatrix} \mu_k > 0$ using the recurrence relation , every term (but μ_{2n+1}) of this inequality is real, therefore μ_{2n+1} has to be real. We have therefore proved that $\mathcal{L}$ has real moments.

.We can now construct explicitly an OPS for $\mathcal{L}$. Let us take $P_0 = \mu_0^{-\frac{1}{2}}$ and for any integer n, $Q_{n+1}(X) = X^{n+1} - \sum_{k=0}^{n} a_k P_k(X)$ where $a_k = \mathcal{L}[X^{n+1} P_k(X)]$ and $P_{n+1}(X) = \left(\mathcal{L}[Q_{n+1}(X)^2]\right)^{-\frac{1}{2}} Q_{n+1}(X)$. Then it is easy to prove by recurrence that P_n is real for any n. Let us prove by strong recurrence that

$$\forall m \leq n, \mathcal{L}[P_m(X) P_n(X)] = \delta_{m,n}$$

For n=1

$$\mathcal{L}[P_0(X)P_1(X)] = \mu_0^{-\frac{1}{2}} \mathcal{L}[(X - a_0\mu_0^{-\frac{1}{2}}) \left(\mathcal{L}[P_0(X)^2]\right)^{-\frac{1}{2}}] = \mu_0^{-\frac{1}{2}} \mathcal{L}[(X - a_0\mu_0^{-\frac{1}{2}})\mu_0^{\frac{1}{2}}]$$

$$= \mathcal{L}[(X - a_0\mu_0^{-\frac{1}{2}})] = \mathcal{L}[X] - a_0\mu_0^{-\frac{1}{2}}\mathcal{L}[1] = \mu_1 - a_0\mu_0^{-\frac{1}{2}}\mu_0 = \mu_1 - a_0\mu_0^{\frac{1}{2}} = \mu_1 - \mathcal{L}[XP_0(X)]\mu_0^{\frac{1}{2}}$$

$$= \mu_1 - \mu_0^{-\frac{1}{2}}\mathcal{L}[X]\mu_0^{\frac{1}{2}} = \mu_1 - \mu_1 = 0$$

and,

$$\mathcal{L}[P_1(X)^2] = \mathcal{L}[(\mathcal{L}[Q_1(X)^2])^{-1} Q_1(X)^2] = 1$$

So the property is true for n=1. Let us suppose that it is true for any k less than or equal to a given n. Then we have the relation

$$\mathcal{L}[P_m(X)P_{n+1}(X)] = \mathcal{L}[P_m(X)(X^{n+1} - \sum_{k=0}^{n} a_k P_k(X))] \left(\mathcal{L}[Q_{n+1}(X)^2]\right)^{-\frac{1}{2}} \tag{3.4}$$

$$= (a_m - \sum_{k=0}^{n} a_k \mathcal{L}[P_m(X)P_k(X)]) \left(\mathcal{L}[Q_{n+1}(X)^2]\right)^{-\frac{1}{2}}$$

.if m<n+1,

(3.4)= $(a_m - \sum_{k=0}^{n} a_k \delta_{m,k}) \left(\mathcal{L}[Q_{n+1}(X)^2]\right)^{-\frac{1}{2}} = (a_m - a_m) \left(\mathcal{L}[Q_{n+1}(X)^2]\right)^{-\frac{1}{2}} = 0$

.if m=n+1,

(3.4)= $\mathcal{L}[P_{n+1}(X)^2] = \mathcal{L}[(\mathcal{L}[Q_{n+1}(X)^2])^{-1} Q_{n+1}(X)^2] = 1$.

This finishes the proof. □

Now let us introduce a Lemma that will be very useful later.

Lemma 3.9.

Hypothesis: *π is a positive polynomial, not identically equal to 0*

Conclusion: $\exists (p,q) \epsilon \mathbb{R}[X], \pi(X) = p(X)^2 + q(X)^2$

Proof. π being positive, its real zeros have an even multiplicity and its complex zeros are self-conjugated. Therefore we can write $\pi(X) = r(X)^2 \prod (X - \alpha_i)(X - \bar{\alpha}_i)$. Let $\prod (X - \alpha_i) = A(X) + iB(X)$ then $\prod (X - \bar{\alpha}_i) = A(X) - iB(X)$. And so $\pi(X) = r(X)^2 (A(X)^2 + B(X)^2) = p(X)^2 + q(X)^2$. This finishes the proof. □

Theorem 3.10.

Hypothesis: *$\mathcal{L}$ is a moment functional*

Conclusion: *$\mathcal{L}$ is positive-definite$\Longleftrightarrow$(its moments are real and $\Delta_n > 0$ for any n)*

Proof. .Because $\Delta_n > 0$, we have $\Delta_n \neq 0$, therefore by the Theorem 3.2, there exists a unique monic OPS for $\mathcal{L}$ let us call it (P_n). Then by Theorem 3.6 and because (P_n) is monic we have $\mathcal{L}[P_n(X)^2] = \frac{\Delta_n}{\Delta_{n-1}} > 0$. Let $p \epsilon \mathbb{R}[X]$, because of their degree, the elements of the OPS form a basis for $\mathbb{R}[X]$. Therefore we can write $p(X) = \sum_{k=0}^{n} a_k P_k(X)$, and so

$$\mathcal{L}[p(X)^2] = \sum_{j=0}^{n} \sum_{k=0}^{n} a_k a_j \mathcal{L}[P_k(X)P_j(X)] = \sum_{k=0}^{n} a_k^2 L[P_k(X)^2] > 0$$

Let π be a positive not identically 0 polynomial. By Lemma 3.9,

$$\exists (p,q) \epsilon \mathbb{R}[X], \pi(X) = p(X)^2 + q(X)^2$$

And therefore $\mathcal{L}[\pi(X)] = \mathcal{L}[p(X)^2] + \mathcal{L}[q(X)^2] > 0$ so $\mathcal{L}$ is positive-definite.

.Conversely, if $\mathcal{L}$ is positive-definite, then by the Theorem 3.8, its moments are real and it has a real monic OPS. We have $\mathcal{L}[P_n(X)^2] > 0$ because $\mathcal{L}$ is positive-defnite, but Theorem 3.6 says $\mathcal{L}[P_n(X)^2] = \frac{\Delta_n}{\Delta_{n-1}}$ therefore all the Δ_n must have the same sign. But $\Delta_0 = \mu_0 > 0$ (result seen in the proof of Theorem 3.8) therefore all the Δ_n are strictly positive. This finishes the proof. □

4. Favard's Theorem

Theorem 4.1.

Hypothesis:$\mathcal{L}$ *be a moment functional, and* (P_n)*be its monic OPS.*

Conclusion: P_n *satisfies the following three-terms recurrence relation:*

$$\begin{cases} P_{n+1}(X) = (X - c_n)P_n(X) - \lambda_n P_{n-1}(X) \\ P_0(X) = 1 \\ P_{-1}(X) = 0 \end{cases}$$

where $c_n = \mathcal{L}[XP_n(X)^2]$ *and* $\lambda_n = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$.

Proof. Because P_{n+1} is monic and because $\{P_0, \cdots, P_n\}$ form a basis of $\mathbb{R}_n[X]$, we can write :

$$P_{n+1}(X) = (X - c_n)P_n(X) - \lambda_n P_{n-1}(X) + \sum_{k=0}^{n-2} c_k P_k(X) \tag{4.1}$$

Then applying the operator $\mathcal{L}[P_k.\square]$ to (4.1) we obtain for $k < n-1$:

$$\mathcal{L}[P_k(X).(XP_n(X))] + c_k \mathcal{L}[P_k(X)^2] = 0$$

i.e

$$\mathcal{L}[(XP_k(X)).P_n(X)] + c_k \mathcal{L}[P_k(X)^2] = 0$$

but $\deg(XP_k(X)) < n$ therefore $\mathcal{L}[(XP_k(X)).P_n(X)] = 0$ moreover, $\mathcal{L}[P_k(X)^2]$being non-zero, we obtain $c_k = 0$.

We can now apply the operator $\mathcal{L}[P_{n-1}.\square]$ to (4.1) which leads to

$$0 = \mathcal{L}[(XP_{n-1}(X)).P_n(X)] - \lambda_n \mathcal{L}[P_{n-1}(X)^2] = \mathcal{L}[X^n.P_n(X)] - \lambda_n \mathcal{L}[P_{n-1}(X)^2]$$

$$= \mathcal{L}[P_n(X)^2] - \lambda_n \mathcal{L}[P_{n-1}(X)^2]$$

therefore by theorem 3.6

$$0 = \frac{\Delta_n}{\Delta_{n-1}} - \lambda_n \frac{\Delta_{n-1}}{\Delta_{n-2}}$$

and so $\lambda_n = \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$.

We can now apply the operator $\mathcal{L}[P_n.\square]$ to (4.1) which leads to

$$0 = \mathcal{L}[XP_n(X)^2] - c_n \mathcal{L}[P_n(X)^2]$$

and therefore $c_n = \frac{\mathcal{L}[XP_n(X)^2]}{\mathcal{L}[P_n(X)^2]}$. This finishes the proof. □

So we have seen that, given a moment functional $\mathcal{L}$, the associated OPS automatically verifies a three-terms recurrence relation. There is a beautiful converse to this fact, which is known as Favard's theorem, and that says that every polynomial family, which verifies such a three term recurrence relation, is in fact an OPS for a certain moment functional.

Definition 4.2. A moment functional $\mathcal{L}$ is said quasi-defnite $\iff \forall n \epsilon \mathbb{N}, \Delta_n \neq 0$

Theorem 4.3. Favard's Theorem

- ***Hypothesis***:
  - $(c_n)\,,(\lambda_n)\,\epsilon\mathbb{C}^{\mathbb{N}}$
  - *The polynomial family* (P_n)*is defined by the 3 terms recurrence relation*
    $$\begin{cases} P_n(X) = (X - c_n)P_{n-1}(X) - \lambda_n P_{n-2}(X) \\ P_{-1}(X) = 0 \\ P_0(X) = 1 \end{cases}$$

- ***Conclusion:***
  (1) *There exists a unique moment functional* $\mathcal{L}$ *such that* $\mathcal{L}[1] = \lambda_1$ *and* $\mathcal{L}[P_m(X)P_n(X)] = 0$ *for* $m \neq n$
  (2) $\mathcal{L}$ *is quasi-definite and* (P_n) *is its monic OPS*$\Longleftrightarrow \forall n\epsilon\mathbb{N}, \lambda_n \neq 0$
  (3) $\mathcal{L}$ *is positive-definite* $\Longleftrightarrow\forall n\epsilon\mathbb{N}, \lambda_n > 0$ *and* c_n *is real*

Proof. (1). Let us construct $\mathcal{L}$ explicitly by the following recursive definition.
$\mathcal{L}[1] = \lambda_1 = \mu_0$ then we use the fact that we want for any n>0 $\mathcal{L}[P_n(X)] = 0$ and we use the recurrence relation to define (μ_n). For n=1, we have

$$P_1(X) = (X - c_1)P_0(X) - \lambda_1 P_{-1}(X) = X - c_1 = 0$$

therefore $\mu_1 - c_1\mu_0 = 0$ and we have found μ_1. Then the μ_n are uniquely determined. So, for a given μ_0, we have a unique $\mathcal{L}$ that verifies $\mathcal{L}[P_n(X)] = 0$. Then we can prove by recurrence that $\mathcal{L}[X^m P_n(X)] = 0$ for any n and any m<n. The recurrence relation can be written

$$P_{n+1}(X) + cP_n(X) + \lambda P_{n-1}(X) = XP_n(X) \tag{4.2}$$

If we apply $\mathcal{L}$ to (4.2), we obtain

$$\mathcal{L}[XP_n(X)] = \mathcal{L}[P_{n+1}(X)] + c_n\mathcal{L}[P_n(X)] + \lambda_n\mathcal{L}[P_{n-1}(X)] = 0$$

by construction of $\mathcal{L}$. So for any n>1, $\mathcal{L}[XP_n(X)] = 0$. If we multiply the identity (4.2) by X then apply $\mathcal{L}$, we obtain:

$$\mathcal{L}[X^2P_n(X)] = \mathcal{L}[XP_{n+1}(X)] + c_n\mathcal{L}[XP_n(X)] + \lambda_n\mathcal{L}[XP_{n-1}(X)]$$

So for any n>2, $\mathcal{L}[X^2P_n(X)] = 0$. Then, if we keep going this way, we obtain for any m, for any n>m, $\mathcal{L}[X^mP_n(X)] = 0$. Therefore there exists a unique moment functional $\mathcal{L}$ such that $\mathcal{L}[1] = \lambda_1$ and $\mathcal{L}[P_m(X)P_n(X)] = 0$ for $m \neq n$. This proves the property (1).

(2).For m=n, we have

$$\mathcal{L}[X^nP_n(X)] = \mathcal{L}[X^{n-1}P_{n+1}(X)] + c_n\mathcal{L}[X^{n-1}P_n(X)] + \lambda_n\mathcal{L}[X^{n-1}P_{n-1}(X)]$$

$$= \lambda_n\mathcal{L}[X^{n-1}P_{n-1}(X)]$$

Therefore,

$$\mathcal{L}[X^nP_n(X)] = \lambda_n\lambda_{n-1}\cdots\lambda_2\lambda_1 \tag{4.3}$$

.So if $\forall n\epsilon\mathbb{N}, \lambda_n \neq 0$, then $\mathcal{L}[P_n(X)^2] \neq 0$ which was the missing property to ensure that (P_n) is an OPS (monic because of the recurrence relation) for $\mathcal{L}$ (with Lemma 3.1) so by Theorem 3.5, $\Delta_n \neq 0$ which means that $\mathcal{L}$ is quasi-definite.

.Conversely, if $\mathcal{L}$ is quasi-definite and (P_n) is its monic OPS, then $\mathcal{L}[P_n(X)^2] \neq 0$ which leads directly (with (4.3)) to the fact that $\forall n\epsilon\mathbb{N}, \lambda_n \neq 0$. This proves the property (2).

(3).**.**If $\forall n \epsilon \mathbb{N}, \lambda_n > 0$ and c_n is real, then all the μ_n are real (by construction and because c_n and λ_n are real) and by Theorem 3.6. We have the relation:

$$\mathcal{L}[P_n(X)^2] = \frac{\Delta_n}{\Delta_{n-1}} = \lambda_n \lambda_{n-1} \cdots \lambda_2 \lambda_1 > 0$$

therefore all the Δ_n have the same sign but $\Delta_0 = \mu_0 > 0$ so $\forall n \epsilon \mathbb{N}, \Delta_n > 0$. And therefore, by Theorem2.10, $\mathcal{L}$ is positive-definite.

.Conversely, if $\mathcal{L}$ is positive-definite then by Theorem 3.10 $\forall n \epsilon \mathbb{N}, \Delta_n > 0$ and then $\mathcal{L}[P_n(X)^2] = \frac{\Delta_n}{\Delta_{n-1}} = \lambda_n \lambda_{n-1} \cdots \lambda_2 \lambda_1 > 0$ and this is true for any n, this means that all the λ_n are strictly positive. Moreover, the $\mu_n$and the λ_n being real, their construction implies that the c_n are real as well. This proves the property (3) and finishes the proof of the theorem. □

Part 2. **Complex generalization and link to Julia sets**

5. Introduction to Julia sets[2]

Definition 5.1. In numerical analysis, the ***fixed point iteration method*** is a method of computing fixed points of iterated function. More precisely, given a continuous function $f : \mathbb{C} \longrightarrow \mathbb{C}$ and an initial value z^0, we create the following sequence: $\begin{cases} z_{n+1} = f(z_n) \\ z_0 = z^0 \end{cases}$. If this sequence converges, then the limit is a fixed point of f.

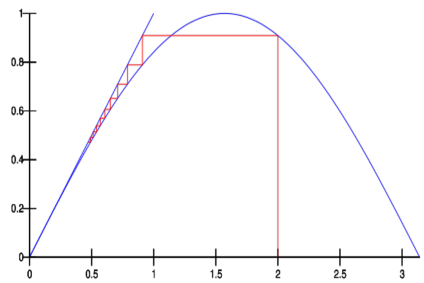

Figure 5.1. Example o fthe fixed point iteration method for the function sinus with initial value $z_0 = 2$

We will now focused our attention on the fixed point iteration method obtained for a polynomial T.

Definition 5.2. A point $z \epsilon \mathbb{C}$ is called an attractive point for T, if there exist an initial condition $z_0 \epsilon \mathbb{C}$ such that the sequence defined by the related fixed point iteration method converges to z.

Remark 5.3. z can be finite or infinite.

Definition 5.4. We call the ***basin of attraction*** of an attractive point z,and we note $A_T(z)$, the set of all the initial conditions z_0 such that sequence defined by the related fixed point iteration method converges to z i.e $\mathrm{A}_T(z) = \left\{ z_0 \epsilon \mathbb{C}, \lim_{n \to \infty} z_n = z \right\}$

Example 5.5. Let us consider the polynomial $T(z) = z^2$, then $A_T(1) = \{-1, 1\}$

Definition 5.6. The ***polynomial Julia set*** J_T set is the boundary of the basin of attraction of the point at infinity i.e $J_T = \partial A_T(\infty)$

Example 5.7. Let consider the polynomial $T(z) = z^2$ then
$z_n = z_0^{2n}$ so $\lim_{n \to \infty} z_n = \infty \Longleftrightarrow |z_0| > 1$.

Let D_1 be the unit disc of $\mathbb{C}$ and $C_1 = \partial D_1$ be the unit circle. Therefore $A_T(\infty)$=$\mathbb{C} \diagdown D_1$ and $J_T = C_1$.

[2]A nice introduction, historically and mathematically complete can be found in [7]

Note 5.8. In the previous example, the Julia Set associated to the polynomial T is very simple. Nevertheless, these sets are most of the time fractal sets. For example, let us consider the Polynomial $T(z) = z^2 + c$ where c is a complex number, which is the classical example that leads to fractal sets. Here are some pictures[3] of the Julia sets obtained for different values of c.

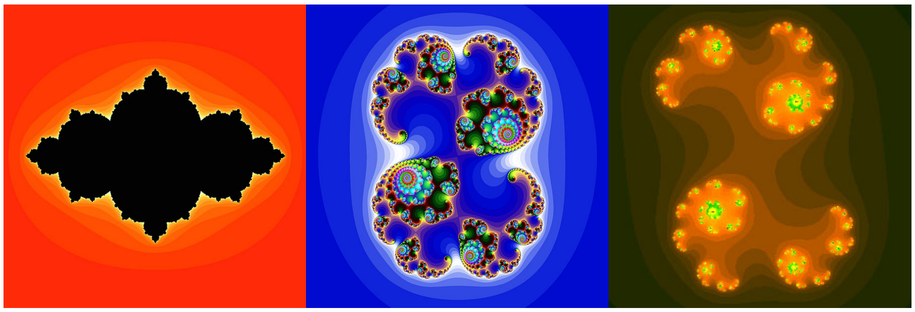

FIGURE 5.2. $c = 1 - \frac{1+\sqrt{5}}{2}$; $c = 0.285 + 0.01i$; $c = 0.45 + 0.1428i$

6. Reminder of complex analysis[4]

Definition 6.1. Let U be an open set of $\mathbb{C}$ and $f : U \longrightarrow \mathbb{C}$. We say that f is ***complex differentiable*** at the point $z_0 \epsilon \mathbb{C}$ if $\lim_{z \longrightarrow z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. If it is the case, this limit is noted $\frac{df}{dz}(z_0)$.

Definition 6.2. We say that a function f is ***holomorphic*** on an open subset $U \subset \mathbb{C}$, if f is complex differentiable at any $z_0 \epsilon U$.

Definition 6.3. Given a closed complex contour Γ and a function f, parametrized by a function φ piecewise differentiable on $[a, b]$, then we define ***the integral of** f **on the contour*** Γ by the following relation: $\oint_\Gamma f(z)dz = \int_a^b f(\varphi(t))\varphi'(t)dt$

Definition 6.4. We say that a compact set $K \subset \mathbb{C}$ is ***regular***, if ∂K is a piecewise differentiable curve.

Theorem 6.5. ***Cauchy's Formula***

Hypothesis:

- *f is holomorphic on an open subset $U \subset \mathbb{C}$*
- *K is a regular compact of U*
- $z_0 \epsilon K \diagdown \partial K$

Conclusion: $f(z_0) = \frac{1}{2i\pi} \oint_{\partial K} \frac{f(z)}{z-z_0} dz$

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[3]All the pictures of Julia sets of this essay have been taken from the article on Julia sets of Wikipedia.

[4]These results are taken from my undergraduate course in complex analysis: [8]. Because the results are very classic, I took the responsability not to prove all of them.

Definition 6.6. We say that a function f is ***analytic*** at a point $a\epsilon\mathbb{C}$, if there exists a disc centered in a (noted $D(a)$) such that $\exists(c_n)\epsilon\mathbb{C}^{\mathbb{N}}$, $\forall z\epsilon D(a)$, $f(z) = \sum\limits_{n=0}^{\infty} c_n(z-a)^n$.

Theorem 6.7. *A function f is analytic at a point $a\epsilon\mathbb{C}\Longleftrightarrow\exists U\subset\mathbb{C}$ open, such that $a\epsilon U$ and f is holomorphic on U.*

Proof. **.**Let f be holomorphic in a point a, then by definition we can find an open disc D around a such that f is holomorphic there. Let us pick a point, say z, inside this disc and then consider a circle C that lies inside the disc D such that z is in the interior of the circle C. Then the Cauchy's formula says :

(6.1)

$$f(z) = \frac{1}{2i\pi}\oint\limits_C \frac{f(w)}{w-z}dw = \frac{1}{2i\pi}\oint\limits_C \frac{w-a}{w-a}\frac{f(w)}{w-a-(z-a)}dw = \frac{1}{2i\pi}\oint\limits_C \frac{1}{w-a}\frac{f(w)}{1-\frac{z-a}{w-a}}dw$$

But $\left|\frac{z-a}{w-a}\right| < 1$ by construction. Therefore $\frac{1}{1-\frac{z-a}{w-a}} = \sum\limits_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$ and the series converges uniformly, therefore it commutes with the integral. Then (6.1) becomes:

$$\sum_{n=0}^{\infty}(z-a)^n\frac{1}{2i\pi}\oint\limits_C \frac{f(w)}{(w-a)^{n+1}}dw = \sum_{n=0}^{\infty}c_n(z-a)^n$$

And this is true for any z in the open disc generated by C, therefore f is analytic in a.

.If f is analytic in a , we can write for any z in an open set containing a, that $f(z) = \sum\limits_{n=0}^{\infty} c_n(z-a)^n$. Such a series is well known to be infinitely differentiable in a therefore it is differentiable once and then f is holomorphic in a. This finishes the proof. □

Theorem 6.8.

Hypothesis: *Let f be an analytic function on an annulus*
$A = \{z\epsilon\mathbb{C}, 0 < R_1 \leq |z| \leq R_2\}$ *and* Γ_r *any circle define by* $|z| = r$ *and* $R_1 < r < R_2$

Conclusion: *Then f has a **Laurent series** development on A i.e*
$\forall z\epsilon A, f(z) = \sum\limits_{n=-\infty}^{\infty}a_n z^n$ *and* $a_n = \frac{1}{2i\pi}\oint_{\Gamma_r}\frac{f(z)}{z^{n+1}}dz$

7. Padé Approximation[5]

Definition 7.1. Let g be an analytic function at the origin of the complex plane. A Padé approximant to g is a rational function of degree (n,m) noted $[n,m]_g$such that $g(z) - [n,m]_g(z) \underset{z\to 0}{=} O(|z|^{m+n+1})$. In other terms , we can write :

$$[n,m]_g(z) = \sum_{n=0}^{m+n}\frac{z^n}{n!}g^{(n)}(0) + O(|z|^{m+n+1})$$

[5]More about this topics in [3]or [6]

If g is analytic in a neighbourhood of infinity, then the Padé approximant of degree (n,m) to g at infinity, noted $[n,m]_g^{\infty}$ verify:

$$g(z) - [n,m]_g^{\infty}(z) \underset{z\to\infty}{=} O(\left|\frac{1}{z}\right|^{m+n+2})$$

Proposition 7.2. *Given an analytic function g in a neighbourhood of infinity, the Padé approximant of a given degree $(n,m), n \leq m$ is unique.*

Proof. Let $R_1 = \frac{P_1}{Q_1}$ and $R_2 = \frac{P_2}{Q_2}$ be two Padé approximant to g of degree (n,m) Then we have the two following relations: $\begin{cases} g(z) - R_1(z) = O(\left|\frac{1}{z}\right|^{m+n+2}) \\ g(z) - R_2(z) = O(\left|\frac{1}{z}\right|^{m+n+2}) \end{cases}$ by subtracting those two relations we obtain: $R_1(z) - R_2(z) = O(\left|\frac{1}{z}\right|^{m+n+2})$ i.e $\frac{P_1(z)}{Q_1(z)} - \frac{P_2(z)}{Q_2(z)} = O(\left|\frac{1}{z}\right|^{m+n+2})$ so by multiplying by Q_1Q_2 it follows

$$Q_2(z)P_1(z) - Q_1(z)P_2(z) = O(\left|\frac{1}{z}\right|^{m-n+2})$$

but the right hand side of this expression is a polynomial of degree $m+n$, and $m-n+2 > 0$ this force the polynomial to tend to zero when z tends to infinity, which is equivalent to the fact that the polynomial is equal to zero everywhere. Therefore $Q_2(z)P_1(z) - Q_1(z)P_2(z) = 0$ and so by dividing by Q_1Q_2 it follows that $R_1 = R_2$. Here we can divide easily as the region we are interested in is a neighbourhood of infinity, therefore we can choose it outside the poles of R_1 and R_2. □

8. Orthogonal polynomials on a complex contour[6]

Definition 8.1. Let g be an analytic function in a neighbourhood of infinity, that can be written $g(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$ [7]. Let Γ be a big enough circle to be in the analycity domain of g. We say that a family of complex polynomials (P_n) is an OPS with respect to the weight function g on the contour Γ , if:

$$\exists (h_n)\epsilon(\mathbb{C}^*)^{\mathbb{N}}, s.t., \forall (n,m)\epsilon\mathbb{N}^2, \frac{1}{2i\pi}\oint_{\Gamma} P_n(z)P_m(z)g(z)dz = h_n\delta_{m,n}$$

Note 8.2. As it can be noticed, in the complex we have decided to work with the first definition of an OPS given at the begining of the first Part (Definition 2.7). The reason why we do not use the other formal definition that deals with moment functional, is to avoid entering into deep measure theory. We will assume those 2 definitions to be equivalent and therefore, to avoid repetitions in this essay, we will now assume (since the proofs are similar) that the results proved in the real

[6]The results presented in section 8 and 9 are mainly taken from [2],[4]or [5], but the proofs are more detailed.

[7]g is therefore analytic in any annulus with a lower circle that lies in the analycity domain of g. So we can develop g in Laurent series by Theorem 6.8 i.e. $g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ with $a_n = \frac{1}{2i\pi}\oint_{\Gamma_r} \frac{g(z)}{z^{n+1}}dz$ so, if we impose that g can be written $g(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$, it is the same that to impose that $g(z) \underset{|z|\to\infty}{\longrightarrow} 0$.

case are conserved in the complex case. Especially the fact that for a given weight function g and a contour Γ, there exists a unique monic OPS w.r.t g on Γ and that all OPS verify the three-terms recurrence relation. Actually, not everything is conserved. For example, the results about the location of the zeros of the orthogonal polynomials are not. This is why I chose not to talk about it in my first part.

Remark 8.3. When we write, $g(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$, the μ_ns are going to play the role of what we have called before the "moments". In fact,

$$\frac{1}{2i\pi}\oint_\Gamma z^k g(z)dz = \frac{1}{2i\pi}\oint_\Gamma z^k \sum_{n=0}^{\infty}\frac{\mu_n}{z^{n+1}}dz = \frac{1}{2i\pi}\oint_\Gamma z^k g(z)dz = \sum_{n=0}^{\infty}\frac{\mu_n}{2i\pi}\oint_\Gamma \frac{dz}{z^{n-k+1}}$$

$$= \sum_{n=0}^{\infty}\frac{\mu_n}{2i\pi}\int_0^{2\pi}\frac{iRe^{i\theta}d\theta}{(Re^{i\theta})^{n-k+1}} = \sum_{n=0}^{\infty}\frac{\mu_n}{2\pi R^{n-k}}\int_0^{2\pi} e^{-i(n-k)\theta}d\theta = \sum_{n=0}^{\infty}\frac{\mu_n}{2\pi R^{n-k}}2\pi\delta_{n,k} = \mu_k$$

Note 8.4. When the circle Γ is the unit circle, the polynomial of a corresponding OPS are called Szègö polynomials.

Theorem 8.5.

- ***Hypothesis:***
  - *g is an analytic function in the neighbourhood of infinity*
  - *Γ is a circle included in the analycity domain of g*
- ***Conclusion:*** *The constituant monic polynomials P_n of the OPS associated to $(\Gamma, g)$are the denominator of the Padé approximation $[n-1/n]_g^{\infty}$.*

Proof. Let $n\epsilon\mathbb{N}$. We can, without loss of generality, suppose that the poles of $[n-1/n]_g^{\infty}$ are in the interior of the circle Γ. Therefore, $[n-1/n]_g^{\infty}(z)$ is holomorphic for $|z| \geq R_\Gamma$ where R_Γ is the radius of the circle Γ. Therefore, the function $z \longmapsto z^{2n+1}(g(z) - [n-1/n]_g^{\infty}(z))$ is holomorphic for $|z| \geq R_\Gamma$. Let call this function $R_n(z)$. But we know that $g(z) - [n-1/n]_g^{\infty}(z) = O(\left|\frac{1}{z}\right|^{2n+1})$. Therefore there exists a constant c such that $|R_n(z)| < c$ for large enough z. And $D_n(z) \underset{z\to\infty}{\sim} z^n$, therefore for large enough z, $|D_n(z)| \leq 2\,|z|^n$. If we write $[n-1/n]_g^{\infty}(z) = \frac{N_{n-1}(z)}{D_n(z)}$ where $N_{n-1}(z)$ is a polynomial of degree n-1 and $D_n(z)$ is a monic polynomial of degree n, we have

$$g(z)D_n(z) - N_{n-1}(z) = R_n(z)\frac{D_n(z)}{z^{2n+1}}$$

Let $k\epsilon\{0, 1, \cdots, n-1\}$. If we multiply the previous relation by $z^k$and then integrate along Γ, we obtain:

$$\frac{1}{2i\pi}\oint_\Gamma z^k g(z)D_n(z)dz - \frac{1}{2i\pi}\oint_\Gamma z^k N_{n-1}(z)dz = \frac{1}{2i\pi}\oint_\Gamma R_n(z)\frac{D_n(z)}{z^{2n+1-k}}dz$$

The second term of the left hand side of the previous equation is equal to 0. In fact, if we note, $f(z) = z^{k+1}N_{n-1}(z)$, f is holomorphic on $\mathbb{C}$, and Γ can be considered as the boundary of a regular compact of $\mathbb{C}$ (the closed disc of radius R_Γ). 0 is in the interior of this compact, therefore we can apply the Cauchy's formula:

$$f(0) = \frac{1}{2i\pi}\oint_\Gamma \frac{f(z)}{z}dz = \frac{1}{2i\pi}\oint_\Gamma z^k N_{n-1}(z)dz$$

But it is obvious that $f(0) = 0$, therefore the result follows.

.The term of the right hand side of the equation is also equal to 0. In fact,

$$\left| \frac{1}{2i\pi} \oint_{\Gamma} R_n(z) \frac{D_n(z)}{z^{2n+1-k}} dz \right| = \left| \frac{1}{2i\pi} \int_0^{2\pi} R_n(R_\Gamma e^{i\theta}) \frac{D_n(R_\Gamma e^{i\theta})}{(R_\Gamma e^{i\theta})^{2n+1-k}} i R_\Gamma e^{i\theta} d\theta \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \left| R_n(R_\Gamma e^{i\theta}) \right| \left| D_n(R_\Gamma e^{i\theta}) \right| R_\Gamma^{-2n+k} d\theta$$

$$\leq c R_\Gamma^{-2n+k} \sup_{\theta \epsilon [0,2\pi]} \left| D_n(R_\Gamma e^{i\theta}) \right| \leq 2c R_\Gamma^{-2n+k} R^n \leq 2c R_\Gamma^{k-n}$$

and by Theorem 6.8 , the left hand side of this inequality is constant. But the right hand side $\underset{R_\Gamma \to \infty}{\longrightarrow} 0$ for $k < n$ therefore the left hand side is equal to 0.

.If $k = n$, we have that for the same reason the second term of the right hand side is equal to zero. But

$$\frac{1}{2i\pi} \oint_{\Gamma} R_n(z) \frac{D_n(v)}{z^{2n+1-k}} dz = \frac{1}{2i\pi} \oint_{\Gamma} R_n(z) \frac{D_n(v)}{z^{n+1}} dz \underset{z \to \infty}{\sim} \frac{1}{2i\pi} \oint_{\Gamma} \frac{R_n(z)}{z} dz$$

because D_n is monic and its degree is n. But by definition $R_n(z) \underset{z \to \infty}{\longrightarrow} \gamma_n$ with $\gamma_n \neq 0$. Therefore

$$\frac{1}{2i\pi} \oint_{\Gamma} \frac{R_n(z)}{z} dz \underset{z \to \infty}{\sim} \frac{\gamma_n}{2i\pi} \oint_{\Gamma} \frac{1}{z} dz = \gamma_n \neq 0$$

.It means that for any $k < n$, $\frac{1}{2i\pi} \oint_\Gamma z^k g(z) D_n(z) dz = 0$, and $\frac{1}{2i\pi} \oint_\Gamma z^n g(z) D_n(z) dz \neq 0$, which is exactly the condition of being an OPS according to the Lemma 3.1. □

9. Topic on Polynomial Iteration

Theorem 9.1.

Hypothesis: *Let W and T be monic polynomials of respective degree d-1 and d*

Conclusion: *The functional equation:* $g(z) = W(z) g(T(z))$ *admit a unique solution* g_∞ *which is holomorphic on the basin of attraction of T at* ∞*:* $A_T(\infty)$ *and respect* $z g(z) \underset{z \to \infty}{\longrightarrow} 1$*.*

Remark 9.2. Since $A_T(\infty)$ is a neighbourhood of infinity, the theorem implies that the solution is holomorphic in a neighbourhood of infinity.

Notation 9.3. We will note the space of the holomorphic functions on a point a, $H(a)$. And the set of the functions holomorphic on a set A, $H(A)$. Therefore the set of the holomorphic functions in a neighbourhood of infinity will be $H(\infty)$.

Note 9.4. This functional equation can be compared with the general concept of Schröder's equation[8] which consists in the following: given p a function that maps the unit disc U to itself and that is holomorphic on it, and a complex number a , we want a holomorphic solution on the unit disc, f , that verifies $\forall z \epsilon U, f(p(z)) = a f(z)$ In fact, if we write our equation in this form (assuming that it is possible...) $\frac{1}{W(z)} g(z) = g(T(z))$, we can clearly see the similitude between the 2 equations, the

[8]More about this topic in [9].

latter being a sort of generalisation from a complex number a to a given rational function. The Shröder's equations are especially important in complex dynamics.

Proof. Let take $g_0 \epsilon H(\infty)$ i.e. $\exists R > 0, \forall |z| > R, g_0(z) = \sum_{k=0}^{\infty} \frac{\mu_k^{(0)}}{z^{k+1}}$. And then construct the following "fixed point iteration method": $g_{n+1}(z) = W(z)g_n(T(z))$ then by iterating it, we get:

$$g_n(z) = W(z)W(T(z))W(T^{(2)}(z))\cdots W(T^{(n-1)}(z))g_0(T^{(n)}(z)) = R_n(z)T^{(n)}(z)g_0(T^{(n)}(z))$$

where

$$R_n(z) = \frac{W(z)W(T(z))W(T^{(2)}(z))\cdots W(T^{(n-1)}(z))}{T^{(n)}(z)}$$

(We can therefore say that $g_n \epsilon H(\infty)$.) T being of degree d, it is obvious that $T^{(n)}$ is of degree d^n. So W being of degree d-1, we have that $W(T^{(k)}(z))$ is of degree $(d-1)d^k$. Therefore

$$deg(W(z)W(T(z))W(T^{(2)}(z))\cdots W(T^{(n-1)}(z))) = \sum_{k=0}^{n-1}(d-1)d^k = d^n - 1$$

Thus R_n is a rational function of degree $(d^n - 1, d^n)$. Because T is monic and of degree d, we have the approximation: $T^{(k)}(z) \underset{z\to\infty}{\backsim} z^{d^k}$. And $W(T^{(k)}(z)) \underset{z\to\infty}{\backsim} (z^{d^k})^{d-1} = z^{d^{k+1}-d^k}$. Therefore

$$R_n(z) \underset{z\to\infty}{\backsim} \frac{\prod_{k=0}^{n-1} z^{d^{k+1}-d^k}}{z^{d^n}} = \frac{z^{d^n-1}}{z^{d^n}} = \frac{1}{z}$$

therefore R_n is holomorphic in a neighbourhood of infinity. But

$$T^{(n)}(z)g_0(T^{(n)}(z)) = \mu_0^{(0)} + O(\frac{1}{|T^{(n)}(z)|}) = \mu_0^{(0)} + O(\frac{1}{|z^{d^n}|})$$

Hence we have:

$$g_n(z) = R_n(z)T^{(n)}(z)g_0(T^{(n)}(z)) = \mu_0^{(0)}R_n(z)+R_n(z)O(\frac{1}{|z^{d^n}|}) = \mu_0^{(0)}R_n(z)+O(\frac{1}{|z^{d^n+1}|})$$

because $R_n(z) \underset{z\to\infty}{\backsim} \frac{1}{z}$ we have that $zg_n(z) \underset{z\to\infty}{\backsim} \mu_0^{(0)}$ so for a limit with the condition $zg(z) \underset{z\to\infty}{\longrightarrow} 1$ to exist, we need $\mu_0^{(0)} = 1$. Therefore $g_n(z) = R_n(z) + O(\frac{1}{|z^{d^n+1}|})$. But R_n , by construction, does not depend on g_0.

Then if the series converges to a $g_\infty \epsilon H(\infty)$, we will have : $\forall n \epsilon \mathbb{N}, \forall k < d^n, \mu_k^{(\infty)} = \mu_k^{(n)}$ therefore, the coefficients being explicitly given by those of R_n. Because we have

$$\left.\begin{array}{l} g_n(z) = R_n(z) + O(\frac{1}{|z^{d^n+1}|}) \\ g_\infty(z) = g_n(z) + O(\frac{1}{|z^{d^n+1}|}) \end{array}\right\} \Rightarrow g_\infty(z) = R_n(z) + O(\frac{1}{|z^{d^n+1}|})$$

We therefore have obtained a Laurent series $g_\infty(z) = \sum_{n=0}^{\infty} \frac{\mu_n^{(\infty)}}{z^{n+1}}$. Let us show that this Laurent series is holomorphe on $A_T(\infty)$.

We know that $zg^\infty(z) \underset{z\to\infty}{\longrightarrow} 1$ by construction. Therefore for a large enough z, $zg^\infty(z)$ can be bounded by a constant, therefore this function is in $H(\infty)$, and it follows that g^∞ is as well (because $z \longrightarrow \frac{1}{z}$ is in $H(\infty)$ and $H(\infty)$ is stable by multiplication). But we have by construction the functional equation: $g^{\infty}(z) = W(z)g^{\infty}(T(z))$ therefore for every point a in $A_T(\infty)$, we can iterate this relation until $T^{(n)}(a)$ enter the analycity domain of g^{∞} and then write

$$g^{\infty}(z) = W(z)W(T(z))\cdots W(T^{(n-1)}(z))g^{\infty}(T^{(n)}(z))$$

then the right hand side is clearly analytic in a therefore g^{∞} is analytic in a and it is true for any a in $A_T(\infty)$, it means that g^{∞} is analytic (holomorphic) on $A_T(\infty)$. This finishes the proof. □

Remark 9.5. Therefore, given the monic polynomial W and T of respective degree d-1 and d,

$$\exists! g \epsilon H(A_T(\infty)), s.t. \begin{cases} \forall z \epsilon \mathbb{C}, g(z) = W(z)T(g(z)) \\ zg(z) \underset{z\to\infty}{\longrightarrow} 1 \end{cases}$$

Theorem 9.6. *If g is the solution found in the previous theorem of the functional equation $g(z) = W(z)T(g(z))$, then the Padé approximants to g at ∞verify the following relation:*

$$[dn-1/dn]_g^{\infty}(z) = W(z)[n-1/n]_g^{\infty}(T(z))$$

Proof. By definition of the Padé approximant, we get $g(z) - [n-1/n]_g^{\infty}(z) = O(\frac{1}{|z|^{2n+1}})$ then, by substituting T(z) for z and multiplying by W(z), we obtain:

$$W(z)g(T(z)) - W(z)[n-1/n]_g^{\infty}(T(z)) = W(z)O(\frac{1}{|T(z)|^{2n+1}}) = O(\frac{1}{|z|^{2nd+1}})$$

i.e. because g is solution of the functional equation,

$$g(z) - W(z)[n-1/n]_g^{\infty}(T(z)) = O(\frac{1}{|z|^{2nd+1}})$$

Therefore, by uniqueness of the Padé approximant, we have

$$[dn-1/dn]_g^{\infty}(z) = W(z)[n-1/n]_g^{\infty}(T(z))$$

□

Theorem 9.7. *Let T be a monic polynomial , then the family of the iterates of T $\left(T^{(n)}\right)$ is a subfamily of an OPS.*

Proof. Let W be a monic polynomial of degree d-1 ($W(z) = z^{d-1} + w_1 z^{d-2} + \cdots$) and T be of degree d ($W(z) = z^d + t_1 z^{d-1} + \cdots$). Let consider g the unique solution in $H(A_T(\infty))$ of

$$\begin{cases} \forall z \epsilon \mathbb{C}, g(z) = W(z)T(g(z)) \\ zg(z) \underset{z\to\infty}{\longrightarrow} 1 \\ \forall a \epsilon A_T(\infty), g(z) = \sum_{k=0}^{\infty} \frac{\mu_k}{z^{k+1}} \end{cases}$$

Then by the previous theorem, we have that

$$[dn-1/dn]_g^{\infty}(z) = W(z)[n-1/n]_g^{\infty}(T(z))$$

if we note $[n-1/n]_g^{\infty}(z) = \frac{Q_{n-1}(z)}{P_n(z)}$ with P_n monic then we have: $P_{dn}(z) = P_n(T(z))$. And then by iterating k times: $P_{d^k n}(z) = P_n(T^{(k)}(z))$ which for n=1 gives $P_{d^k}(z) = P_1(T^{(k)}(z))$. But by theorem 7.2, (P_n) is the monic OPS with respect to g, therefore we have the recurrence relation

$$P_{n+1}(z) = (z - A_n)P_n - R_n P_{n-1}$$

by putting $P_{-1} = 0$, we have $P_1 = (z - A_0)P_0 = z - A_0$. where

$$A_0 = \frac{1}{2i\pi\mu_0}\oint_\Gamma z(P_0(z))^2 g(z)dz = \frac{1}{2i\pi\mu_0}\oint_\Gamma zg(z)dz = \frac{\mu_1}{2i\pi\mu_0}$$

But

$$zg(z) = zW(z)g(T(z)) = (z^d + w_1 z^{d-1} + \cdots)\sum_{k=0}^{\infty}\frac{\mu_k}{T(z)^{k+1}} \underset{z\to\infty}{\backsim} (z^d + w_1 z^{d-1} + \cdots)\frac{\mu_0}{T(z)}$$

$$= (z^d + w_1 z^{d-1} + \cdots)\frac{\mu_0}{z^d + t_1 z^{d-1} + \cdots} = (1 + \frac{w_1}{z} + \cdots)\frac{\mu_0}{1 + \frac{t_1}{z} + \cdots} \underset{z\to\infty}{\backsim} \mu_0(1 + \frac{w_1 - t_1}{z})$$

so by identification we have that $\mu_1 = \mu_0(w_1 - t_1)$ therefore by choosing W such that $w_1 = t_1$, we have $\mu_1 = 0$. And so $A_0 = 0$ which leads to the following relation: $P_{d^k}(z) = T^{(k)}(z)$ and so each iterate of the polynomial T is a particular element of the OPS related to g. This finishes the proof. □

10. Conclusion

We have therefore shown that, given any polynomial T on the complex plane, the family of the iterates of T forms a subfamily of an OPS on any circle that contains the Julia set generated by T.

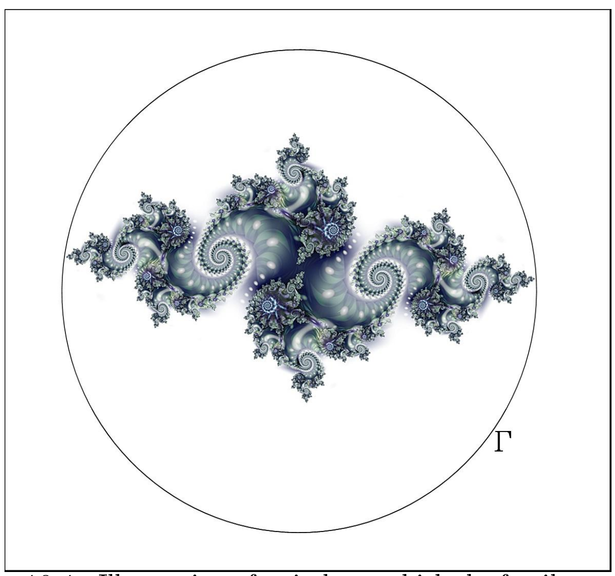

Figure 10.1. Illustration of a circle on which the family generated by the iterates of T is an OPS

This essay could be continued by first trying to reduce the "area" of the contour used in order to be closer to the Julia set itself. For this the restrictive point is the Theorem 6.8. In fact, the contour chosen has to lie in an annulus. If we could extend this theorem to something more general than an annulus, this result would follow. Then we could try to define an integral on a fractal contour and try to discover if the family generated by the iterates of T is actually orhtogonal on the Julia set J_T (when this one is a "closed fractal" line [9] i.e when its connected), it means that we would want to write for an analytic function g:

$$\exists (h_n) \epsilon (\mathbb{C}^*)^{\mathbb{N}}, s.t., \forall (n,m) \epsilon \mathbb{N}^2, \oint_{J_T} P_n(z) P_m(z) g(z) dz = h_n \delta_{m,n}$$

This will certainly lead to further theoretical analysis of the fractal complex contours and will lead to a possible definition of an integral on it.

However, the result that we have shown already has a practical use. In fact, it gives an easy (cheap) way to compute the iterates of a given polynomial (this is usually quite expansive) using the 3 terms recurrence relation that verifies the OPS. Thus the "power" operations (non-linear) are replaced by simple linear additions. This has been shown to be useful in image processing.

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[9] In the case of $T(z) = z^2 + c$ this situation corresponds to the elements c that lie in the interior of the Mandelbrot set.