An Efficient Semi-Analytical Scheme for Determining the Reflection of Lamb Waves in a Semi-Infinite Waveguide

Robert Davey[1,*], **Raphaël Assier**[1], **David Abrahams**[2], **Rich Hewitt**[1]
[1]School of Mathematics, University of Manchester, United Kingdom
[2]Isaac Newton Institute, University of Cambridge, United Kingdom
*Email: robert.davey@postgrad.manchester.ac.uk

Abstract

The reflection of Lamb waves from a free edge perpendicular to an elastodynamic plate is studied. It is known that extant methods for finding the reflected field have poor convergence due to irregular behaviour near corners. The form of the irregularity for an elastodynamic corner is derived asymptotically. A new method for incorporating this form of the corner behaviour is then implemented. Results are presented showing this new method improves convergence in the reflection problem.

Keywords: Elasticity, Lamb Waves, Corner Behaviour, Reflection

1 Introduction

We consider the reflection of an incoming linearly-elastic wave in a semi-infinite elastic waveguide, as shown in Figure 1. The waves associated with two dimensional elastic wave guides were first studied by Lamb almost one hundred years ago [1], and are still an active area of research due to their wide range of applications in nondestructive testing. In addition these waves have interesting mathematical features such as the structure of the dispersion relation [2].

It is well known that the corners present in this model have irregular behaviour, this is caused by having three boundary conditions at the intersection of the free edges. The local behaviour of corners at the intersection of two traction free edges is known to be singular if the angle is greater than π and bounded for angles less than π [3]. In this problem with two corners of angle $\pi/2$, we therefore know the local behaviour is bounded, however this work may also be applied to singular corners.

It has also been known for some time that the solution to our test problem in terms of a modal expansion of Lamb waves has extremely poor convergence [4]. This is due to the irregular behaviour of the corners, similarly to how approximating irregular behaviour by Fourier series results in Gibbs phenomena. In the same way that the convergence of Fourier series can be improved by removing the problematic behaviour, we seek to isolate the irregular corner terms from our Lamb wave expansion. To do this we will introduce new modes that accurately represent the behaviour near the corners and so free the Lamb modes to represent the stress field in the rest of the plate where they provide a useful description.

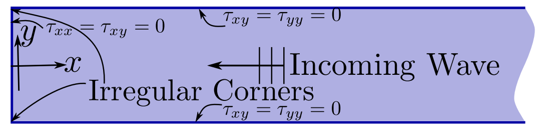


Figure 1: The model we wish to study. There are two corners with locally irregular behaviour at the intersections of the traction free surfaces. A propagating Lamb wave is incoming from the right and we wish to determine the resulting scattered field.

2 Corner Behaviours

We must first find the behaviour that we wish to isolate in our corner modes. To do so we use asymptotic expansions on the solutions of the Navier-Lamé equations in potential form. This enables us to write the corner behaviours as a series of modes. Each mode is known up to a multiplicative constant and has leading order behaviour for all stresses as $\tau \sim \rho^{\gamma_m} - 1$, where ρ is the distance from the corner, γ_m is a solution of the compatibility condition $\sin(\gamma_m \xi) \pm \gamma_m \sin(\xi) = 0$ and $\Re[\gamma_m] > 0$. We now wish to introduce the irregular behaviour of these modes into our expansion.

3 Virtual Plates

In the neighbourhood of the corner, $\rho \ll 1$, we know that the solution will be accurately represented by the corner modes. We can use this in our solution; however, we cannot directly add the corner modes to a modal expansion as they automatically satisfy the boundary conditions. We will instead use our knowledge that the behaviour near the corner in all directions will be dominated by the corner modes, including in an extended 'virtual' domain. As such we know the forms of the behaviour of the stresses on the 'virtual' boundaries $y = \pm 1/2$ for $x < 0$, as shown in Figure 2. Here s, p, q and r are the

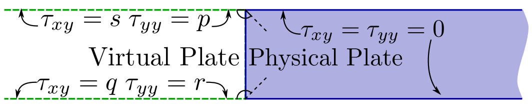


Figure 2: The problem we wish to use to implement corner modes. The shaded physical plate has traction free conditions on top and bottom but the condition on the surface $x = 0$ will be applied later. The dashed lines denote the extended 'virtual' plate where we are imposing the local form of the stresses resulting from the corners.

functions that describe the local behaviour of a corner and are expressed as a series of corner modes.

We can solve this problem by use of Fourier transform methods. In doing so we find that the Fourier transformed stress fields have dependence on each of the corner modes in addition to having poles at the zeroes of the regular dispersion relation for Lamb waves. When taking the inverse Fourier transform we will find that the contribution from the poles yields the well known forms of Lamb waves up to an unknown constant. We will additionally find that the contour, which must be chosen to be consistent with the number of Lamb waves in the expansion, will generate a series of modes that represent the corner dependence. We may write this as a modal expansion given by

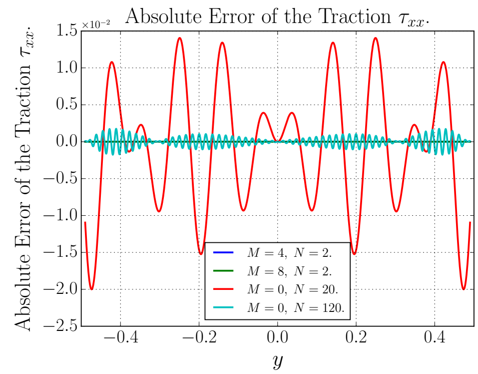
$$\tau_{xx} = \sum_{n=1}^{N} \zeta_n t_x^n(x, y) + \sum_{m=1}^{M} \eta_m s_x^m(x, y), \quad (1)$$

$$\tau_{xy} = \sum_{n=1}^{N} \zeta_n t_y^n(x, y) + \sum_{m=1}^{M} \eta_m s_y^m(x, y), \quad (2)$$

where t^n is the stress field of the nth Lamb mode and s^m is the stress field of the mth corner mode. Here ζ_n and η_m are as yet undetermined constants.

4 Results

We have found that in using the addition modes we have implemented we have much improved convergence, Figure 3 shows that the error in the stress field is smaller when using a small number of corner and Lamb modes when compared to a large number of Lamb modes only.



Text

Figure 3: The absolute errors in the tractions generated by various truncations. In generating these plots the transverse free space wave number of the material was $k_t = 1$ and the Poisson ratio was $\nu = 0.3$.

References

[1] H. Lamb. On waves in an Elastic Plate. *Proceedings of the Royal Society of London*, 1917.

[2] M. L. Williams. Surface singularities resulting from various boundary conditions in angular corners of plates in extension. *Journal of Applied Mechanics, ASME*, 1952.

[3] J. B. Lawrie and J. Kaplunov. Edge waves and resonance on elastic structures: An overview. *Mathematics and Mechanics of Solids*, 2012.

[4] R. D. Gregory and I. Gladwell. The reflection of a symmetric Rayleigh-Lamb wave at the fixed or free edge of a plate. *Journal of Elasticity*, 1983.